Endogenous Coalition of Intellectual Properties: A Three-Patent Story*

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Abstract

We investigate the endogenous coalition formation among intellectual property owners in a three-patent setting, and answer the following questions: (1) What are the profits of patent pools in equilibrium under different pool structures? (2) Under what circumstances is the complete pool or incomplete pool the stable pool structure? (3) Is a market structure of fragmented patents a possible outcome? (4) What is the welfare effect of a stable pool structure? There are two main contributions of this paper. First, it gives a full picture of endogenous pool formation in a tractable framework à la Lerner and Tirole (2004). Particularly, it explores the relationship between pool structure outcome and value accumulation from increasing patents. Second, our analysis is based on the notion of equilibrium binding agreements (Ray and Vohra 1997), and thus provides an intriguing application of theory of coalition formation. *JEL Classification:* (C70, C71, K11, L13, L24).

Keywords: Coalition formation; intellectual property; patent pool

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1 Introduction

Cooperation among intellectual property owners, of which patent pool is a representative, is often observed in a variety of industries. It is estimated that in 2001 sales of devices wholly or partly based on pooled patents exceeded $100 billion (Clarkson 2003). Because of the practical importance of pooling intellectual properties, theoretical literature thereof has been initiated by, among others, Shapiro (2001), Kim (2004) and, remarkably, Lerner and Tirole (2004).

The purpose of this paper is to investigate the endogenous coalition formation among intellectual property owners in a three-patent setting. We mainly answer the following four questions:

(Q1) What are the profits of patent pools in equilibrium under different pool structures?

(Q2) Under what circumstances is the complete pool or incomplete pool the stable pool structure?

(Q3) Is a market structure of fragmented patents a possible outcome?

(Q4) What is the welfare effect of a stable pool structure?

In terms of the basic set-up, our analysis is pursuant to the work of Lerner and Tirole (2004). In their seminal paper, Lerner and Tirole (2004) propose a tractable model of patent pools which facilitates analyzing institutional features and antitrust policy, and provides a necessary and sufficient condition for a complete pool to enhance welfare. There are two interesting features of their model with symmetric patents. First, none of the patents is essential to the technology and each can be used individually or in a collective way. This further introduces the following property of pricing behavior of patent holders in equilibrium. Each patent holder seeks to maximize his profit, but at the same time, with the constraint that the price of patent is low enough to be included in the basket adopted by the users. These two existing forces enrich and complicate the model to a great extent. Since this feature in a sense reflects the cumulative characteristic of innovation (Scotchmer 1991), it is kept and plays an important role in our analysis. Second, the only pooling choice for patent holders is the complete pool. That is to say, in terms of patent pool structures, only polar cases of a grand coalition and all stand-alone patents are considered. All the in-between pool structures are assumed away. Nevertheless, this assumption is restrictive when we study the endogenous formation of pool structures, considering the diversity of pool sizes and structures in practice[1]. Hence, we retrieve the possibility of incomplete pool

[1] An example of absence of complete pool is the Third Generation Patent Platform Partnership (3G3P), founded in 1999, which only provides general rules concerning licensing of patents covering the 3G mobile telecommunication technologies. Five independent PlatformCos (platform companies), each consisting of patents essential to one 3G radio interface technology, implement licensing functions separately (Guellec and de la Potterie 2007).

in our model, and explore when an incomplete pool can be a stable pool structure.[2]

There are two main contributions of our paper. First, it gives a full picture of endogenous coalitional behaviors of intellectual property owners in a tractable framework à la Lerner and Tirole (2004). Particularly, instead of putting an ambiguous tag of substitutability or complementarity, we investigate the very relationship between pool structure outcome and value accumulation from increasing patents. Second, in the course of finding the stable pool structure, we provide a simple but intriguing application of game-theoretic modelling of coalition formation, of which equilibrium binding agreements (Ray and Vohra 1997) is a prominent representative.[3] It is noted that our analysis goes beyond intellectual properties and can be extended to economic phenomena characteristic of similar endogenous pooling arrangements.

The notion of equilibrium binding agreements, proposed by Ray and Vohra (1997), is adopted as a protocol of pool formation in our paper (see Section 4 for its details). Rather than talking about the possibility of pool dissolving in a loose way, we base our analysis on this notion in order to provide a solid strategic foundation in terms of stability of a pool structure. Equilibrium binding agreements is quite involved in its general situation with n heterogeneous players, but intuitive in our three-patent (symmetric) game. Generally speaking, judging by the possibility of internal blocking by farsighted players, this notion first labels every feasible pool structure as being "in equilibrium" or not, and then announces the coarsest one "in equilibrium" to be the stable outcome. The focus on the coarsest "in-equilibrium" pool structure especially makes sense in our setting, seeing that the number of players is only three, and thus facilitates the free negotiation among all of them.

The sophistication of our analysis is twofold. On the one hand, when studying the equilibrium profits given a pool structure, we inherit two elements from the set-up of Lerner and Tirole (2004). As we mentioned above, intellectual property owners, individually or collectively within a pool, try to maximize their profits, however, bearing in mind that their patents could have been evicted from the basket of technology if prices were set too high. (The demand margin is coined to refer to the former element, and the competition margin the latter.) Consequently, as we show in Section 3, to fully characterize the profit allocation under different pool structures, a series of conditions, including the usual dichotomy of concavity and convexity, help and provide some critical cut-off values by the comparison between the value of complete pool and those of smaller pools. In general, the greater the value of complete pool, the more possibly all the pools get rid of the threat of excludability.

[2] For the sake of full tractability, we do our exercise in a setting with three patents, which suffices to cover the case of one incomplete pool.

[3] We believe that the use of this theoretical tool, pitifully rare, broadens our understanding of cooperation among economic agents, especially in the field of industrial organization where cooperative behaviors, e.g., collusions, strategic alliances, joint ventures and mergers, abound. See, for example, Bloch (2002) for a survey of the coalition-formation approach in industrial organization.

In addition, a big pool has more flexibility of pricing than a small pool, since the former contributes more to the complete pool than the latter.

On the other hand, when we carry the set-up above into our analysis of endogenous pool formation, the whole picture becomes even more blurred. In order not to get lost, it is helpful to keep in mind one useful benchmark model: the standard model of output cartels in Cournot oligopoly.[4] When the number of symmetric firms is three, at first sight the complete pool is subject to dissolving, in that one firm has incentive to free-ride given that the other two stick together. However, if we allow for further defections, the other two firms will break apart. This "instability" of incomplete pool makes free-riding unprofitable. Therefore, once firms are regarded as farsighted, as is required by equilibrium binding agreements, the complete pool structure passes the test of stability and becomes the reasonable prediction of structure outcome.[5] Nevertheless, in terms of different combinations of critical conditions, besides the benchmark case above, we have in all seven different cases with various equilibria. In contrast to the benchmark case, due to the pricing advantage of big pool, in many cases (or subcases) the incomplete pool structure is preserved to be "in equilibrium" (and even the stable one), thus retrieving the possibility of free-riding. As a result, a straightforward prediction of structure outcome lacks and the stable pool structure relies on more subtle conditions on the magnitude of complete pool in terms of values of smaller pools. Based on this plethora of results, which are stated in Section 4, a general observation is that, when the value of complete pool is sufficiently high, or that of incomplete pool is sufficiently low, the complete pool is always the stable pool structure. Also, in most cases, the stable pool structure always increases the consumer welfare, as long as the value of complete pool is not too low.

Although all the intellectual properties are symmetric *ex ante* in our model, two kinds of (a)symmetries arising, related to equilibrium profits and pool structures respectively, and their interaction are noteworthy. The presence of threat of being excluded from the basket raises the question of uniqueness of equilibrium profits of patent pools. As we will see below, in one subcase of (symmetric) structure of fragmented patents, there exist one symmetric equilibrium and infinite number of asymmetric equilibria. When focusing on the symmetric equilibrium exclusively, both the complete pool and the (asymmetric) incomplete pool structure may be formed, and moreover, there is no possibility of (symmetric) fragmented pool structure being the stable one. It is also the case when the most asymmetric equilibrium is considered. Nevertheless, in contrast to the polar scenarios, a family of moderately asymmetric equilibria introduce the possibility of (symmetric) fragmented

[4] See, for example, Yi (2003), and Ray and Vohra (1997, 1999). In our setting, it occurs when conditions (a), (d) and (f) hold (see Sections 3 and 4 for details).

[5] This stability may not carry over into the general case of n-firm cartel, which is originally observed by Salant et al. (1983). See Ray and Vohra (1997) for a complete model using equilibrium binding agreements.

structure to be formed eventually. This provides an interesting example in which there is no "monotonicity" of change of stable pool structure with the degree of symmetry and extreme asymmetry does not lead to the finest market structure.[6]

Some other related literature is discussed here. Quint (2014) distinguishes further between essential and nonessential patents, and explores the welfare effect of different kinds of patent pools using a model of logit demand. Still, he does not study the endogenous formation of patent pools. Also based on the model of Lerner and Tirole (2004), Brenner (2009) explores the formation of patent pools in a special class of games, where only one (perhaps incomplete) patent pool is allowed to be formed and a sequential version of unanimity game is adopted as the protocol of pool formation. However, he mainly focuses on the optimal formation rules for preventing welfare decreasing pool equilibria, and does not explicitly characterize the relationship between values of pools of different sizes and categorization of stable pool structures. In this sense, our paper is complementary to his. Aoki and Nagaoka (2006) tackle the problem of endogenous formation of patent pools, by using Maskin's (2003) solution concept in a partition function game. In contrast to Lerner and Tirole (2004), the basic model there on which formation mechanism builds exclusively involves essential patents.

The rest of the paper is organized as follows. Section 2 offers the general set-up and some preliminaries à la Lerner and Tirole (2004). To answer Q1 and provide the building block for our analysis, Section 3 characterizes the equilibrium profits of pools under different pool structures. Section 4 introduces the notion of equilibrium binding agreements as a protocol of pool formation, and presents the main results of our paper. Particularly, we implement a simple algorithm to find the stable pool structure, and provide answers to Q2 and Q4 in a symmetric game. Section 5 extends the discussion of stable pool structure and its welfare effect to the situation with asymmetric equilibria. Especially, a possibility answer to Q3 is provided. Section 6 provides some discussions on the protocol of pool formation and the general n-patent case. Some concluding remarks are given in Section 7. All the proofs are relegated to Appendix B. Several results in the n-patent case are presented in Appendix A.

2 The model and preliminaries

In this section we introduce a general model with n patents à la Lerner and Tirole (2004). There are n intellectual property owners (thereafter, *owners*), each of whom has one intellectual property protected as a patent. A different set of firms (*licensees*) can access some or all of the patents by paying up-front fees (*prices*); i.e., there is no cross-licensing among

[6]The literature of applied theory of coalition formation has normally been confined to symmetric games for the sake of tractability; i.e., given a coalition structure, a player's payoff relies only on the number and the size of coalitions (Section 4 is an example). Section 5 goes beyond the scope of this safe district.

the owners. The cost of patent licensing is zero. Licensees are distributed over $[\underline{\theta}, \overline{\theta}]$, and licensee θ's valuation on the licensed patents is of the form $\theta + V(k)$, where $\theta \in [\underline{\theta}, \overline{\theta}]$, k is the number of patents licensee θ can access, and $V(k)$ is a strictly increasing function of k. In view of this, we say that patents are *symmetric*. Notice that θ can be regarded as the licensee's *private* value regardless of the number of patents to be used, while $V(k)$, not user-specific, is the *common* value. Assume that $\overline{\theta} + V(n) > 0$ and the support $[\underline{\theta}, \overline{\theta}]$ is sufficiently wide to guarantee interior solutions. Let $F(\theta)$ and $f(\theta)$ be the cdf and pdf of θ. The hazard rate function $\frac{f(\theta)}{1-F(\theta)}$ is assumed to be strictly increasing in θ.

A *pool structure* C is a partition of n patents, where each element of C is a (perhaps singleton) pool. Since patents are symmetric, a pool structure C can be concisely written as a set $\{n_1, n_2, ..., n_m\}$, where n_i is the size of pool i, and $\sum_{i=1}^{m} n_i = n$. For any subset $J \subset C$, let $\sharp J \equiv \sum_{n_j \in J} n_j$ denote the number of patents in J.

The prices charged by the pools can be represented by a *price profile* $\boldsymbol{p} \equiv (p_1, p_2, ..., p_m)$, where p_i is the (total) price of n_i patents in pool i for $i = 1, 2, ..., m$. Let $\mathbf{P} \equiv \sum_{j=1}^{m} p_j$, $\mathbf{P}_J \equiv \sum_{n_j \in J} p_j$ for $J \subset C$. Sometimes $\boldsymbol{p}$ is written as $(p_i, \boldsymbol{p}_{-i})$ to emphasize the role of pool i, and $\mathbf{P}_{-i} \equiv \sum_{j \neq i} p_j$.

The timeline of game Γ played by the owners consists of four stages.

Stage 1. The owners form a pool structure $C \equiv \{n_1, n_2, ..., n_m\}$ using some protocol of coalition formation, whose details will be discussed in Section 4.

Stage 2. Given the pool structure C, the price profile $\boldsymbol{p} \equiv (p_1, p_2, ..., p_m)$ is set in the spirit of simultaneous Nash-like play by the pools. We neglect the details of decision-making within each pool, and assume that each pool behaves as if there is a pool administrator who aims to maximize the total profit of owners in this pool. The total profit is assumed to be divided equally within a pool, as patents are symmetric *ex ante*.[7] By contrast, we allow asymmetric equilibria in the sense that the profits of two pools with the same number of patents can be different.

Stage 3. Given the pool structure C and the price profile $\boldsymbol{p}$, each licensee selects the *basket* $B \subseteq C$ based on the following maximization problem[8]

$$\max_{B \subseteq C} \{V(\sharp B) - \mathbf{P}_B\}. \tag{1}$$

Each licensee makes up the basket of some pool(s) in order to maximize the common value (and hence, his total value) net total price of the basket. Clearly, this decision is not user-specific, since it only depends on the common value and the price profile set in stage

[7] In symmetric games, equal intra-coalitional allocation can be endogenously vindicated under some circumstances; see Ray and Vohra (1997, 1999).

[8] To break the tie, we assume that when there exist multiple baskets which are maximizers, the licensees will choose the basket with the largest number of patents.

2.

Stage 4. Given the basket B selected in stage 3, each licensee makes the decision on whether to use the technology comprising all the patents in the basket. Specifically, licensee θ adopts the technology if and only if $\theta + V(\sharp B) \geq \mathbf{P}_B$. These decisions, which depend on the licensees' private values, differ across the licensees.

Given a pool structure C formed in stage 1, we have a subgame $\Gamma(C)$ comprising stages 2-4. In an isomorphic setting of asymmetric patents, Lerner and Tirole's (2004) Proposition 6(i) characterizes the subgame perfect equilibrium of $\Gamma(C)$.[9] Mainly, it indicates three facts related to the equilibrium (see Proposition A1 in Appendix A for its formal presentation):

(1) If some pool has positive sales[10], then all the pools are in the equilibrium basket considering the zero cost of licensing. Therefore, in equilibrium we can focus on the licensees' demand $D(\mathbf{P})$, induced in stage 4, for *all* the patents; i.e.,

$$D(\mathbf{P}) \equiv \Pr(\theta + V(n) \geq \mathbf{P}) = 1 - F(\mathbf{P} - V(n)).$$

(2) In equilibrium each pool is binded by either the *competition margin* or the *demand margin.* Formally, when other pools charge the prices equal to $\boldsymbol{p}_{-i}$, the competition margin $z(\boldsymbol{p}_{-i})$ of pool i is defined as the highest price it can charge; i.e.,

$$V(n) - \mathbf{P}_{-i} - z(\boldsymbol{p}_{-i}) = \max_{J \subseteq C \backslash n_i} \{V(\sharp J) - \mathbf{P}_J\}.$$

Given other pools' prices, if a pool charges a price higher than its competition margin, it will be excluded from the equilibrium basket. Meanwhile, the demand margin of pool i is denoted by

$$r(\boldsymbol{p}_{-i}) \equiv \arg\max_{p} \{pD(p + \mathbf{P}_{-i})\},$$

which is pool i's optimal price in the absence of competition margin.[11] Consequently, pool i will charge the price equal to the minimum of $z(\boldsymbol{p}_{-i})$ and $r(\boldsymbol{p}_{-i})$.

(3) There exists a pool m' such that in equilibrium all the bigger pools, charging the same price, are binded by the demand margin. All the remaining pools are strictly binded by the competition margin.

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[9] As is pointed by Lerner and Tirole (2004) and Brenner (2009), one notorious fact is that there may exist multiple equilibria of $\Gamma(C)$. As we will see in Section 3, even in the simple case of $n = 3$, the number of equilibria can be infinite under some circumstances.

[10] There may exist equlibria with zero sales when each pool charges an unreasonably high price which makes the technology unadoptable for all the licensees even under unilateral price reduction. We ignore these trivial equilibria.

[11] The monotone hazard rate $\frac{f(\theta)}{1-F(\theta)}$ guarantees the quasi-concavity of the profit function and the uniqueness of $r(\boldsymbol{p}_{-i})$ for each $\boldsymbol{p}_{-i}$. Notice that being binded by the demand margin ($r(\boldsymbol{p}_{-i}) \leq z(\boldsymbol{p}_{-i})$) is opposite to being *strictly* binded by the competition margin ($r(\boldsymbol{p}_{-i}) > z(\boldsymbol{p}_{-i})$).

Under a pool structure $C = \{n_1, n_2, ..., n_m\}$, once the equilibrium price profile $\boldsymbol{p} \equiv (p_1, p_2, ..., p_m)$ is determined, the per-owner profit of every pool can be represented by a profit profile

$$\pi(C) \equiv \left(\frac{D(\mathbf{P})p_1}{n_1}, \frac{D(\mathbf{P})p_2}{n_2}, ..., \frac{D(\mathbf{P})p_m}{n_m}\right).$$

The collection of all the profit profiles under feasible pool structures is the building block for our analysis of endogenous coalition of intellectual properties.[12]

3 3-patent case: profits

For notational convenience, let $u \equiv V(1) + \overline{\theta}$, $v \equiv V(2) + \overline{\theta}$ and $x \equiv V(3) + \overline{\theta}$. In the sequel, we consider a special case with $n = 3$. Correspondingly, the only feasible pool structures are $\{1, 1, 1\}$, $\{1, 2\}$ and $\{3\}$ - the fragmented pool structure, an incomplete pool (accompanied by a singleton pool), and one complete pool, respectively. In addition, we make an assumption that licensees are uniformly distributed over $[\overline{\theta} - \Delta, \overline{\theta}]$ where $\Delta \equiv \overline{\theta} - \underline{\theta}$, which induces the linear demand for all three patents in equilibrium. To see this, use $F(\theta) = (\theta - \underline{\theta})/\Delta$ and let $\theta = \mathbf{P} - \left(x - \overline{\theta}\right)$, we have

$$D(\mathbf{P}) = 1 - \frac{\mathbf{P} - \left(x - \overline{\theta}\right) - \underline{\theta}}{\Delta} = \frac{x}{\Delta} - \frac{1}{\Delta}\mathbf{P}.$$

In order to explicitly characterize the profit of every owner, some useful observations (by Proposition A1) are as follows. Consider an equilibrium price profile $\boldsymbol{p}$ under C, and let m be the number of pools, m' and $\mathbf{Z}$ the number and total price of pools strictly binded by the competition margin respectively[13]. Then the pools binded by the demand margin charge the same price as

$$\widehat{p} = \frac{x - \mathbf{Z}}{m - m' + 1}. \tag{2}$$

When $m > 1$, by definition, pool i's demand margin $r(\boldsymbol{p}_{-i})$ satisfies the following

$$r(\boldsymbol{p}_{-i}) = \frac{1}{2}(x - \mathbf{P}_{-i}), \tag{3}$$

[12]Using the jargon of cooperative game theory, the profit profiles can be represented parsimoniously by a partition function $v : \{C | C \text{ is a partition of } n \text{ patents}\} \to \mathbb{R}^n$. Specifically, v assigns to each pool structure C a profit vector $\left(\left(\frac{D(\mathbf{P})p_1}{n_1}\right)_{n_1}, ..., \left(\frac{D(\mathbf{P})p_m}{n_m}\right)_{n_m}\right)$, a configuration of profits for every owner, where $(p_1, ..., p_m)$ is an equilibrium price profile of pools under C.

[13]When $m = 1$ (complete pool), there is no competition margin and $m' = 0$.

and its competition margin $z(\boldsymbol{p}_{-i})$ satisfies the following

$$z(\boldsymbol{p}_{-i}) = V(3) - \mathbf{P}_{-i} - \max_{J \subseteq C \setminus n_i} \{V(\sharp J) - \mathbf{P}_J\}, \tag{4}$$

where, particularly, $J = \{1\}$ ($\{2\}$), if $C = \{1,2\}$ and $n_i = 2$ (1); $J \subseteq \{1,1\}$, if $C = \{1,1,1\}$.

Now we investigate the equilibria under different pool structures. In the case of complete pool, three owners, forming an alliance of intellectual properties, act as a monopoly and split the profit equally. Specifically, the total price $\mathbf{P} = \frac{1}{2}x$ by (2), the licensees' demand $D(\mathbf{P}) = \frac{1}{2\Delta}x$, and the per-owner profit

$$\pi(\{3\}) = \frac{1}{12\Delta}x^2.$$

When the pool structure is $\{1,2\}$ and $\{1,1,1\}$, the equilibria are complicated by the conflicting forces of demand and competition margins. Therefore, the profit profile is contingent on the magnitude difference among x, u and v. Propositions 1 and 2 give a bestiary of profit profiles in the case of incomplete pool and fragmented structure, respectively.

Proposition 1.

$$\pi(\{1,2\}) = \begin{cases} \left(\frac{1}{9\Delta}x^2, \frac{1}{18\Delta}x^2\right), & \textit{if (a): } x \geq \frac{3}{2}v; \\ \left(\frac{1}{2\Delta}v(x-v), \frac{1}{8\Delta}v^2\right), & \textit{if (b): } u + \frac{1}{2}v \leq x < \frac{3}{2}v; \\ \left(\frac{1}{\Delta}(v+u-x)(x-v), \frac{1}{2\Delta}(v+u-x)(x-u)\right), & \textit{if (c): } x < u + \frac{1}{2}v. \end{cases}$$

The categorization of profit profiles does not depend on the dichotomy of *(strict) concavity* ($x - v < v - u$) and *convexity* ($x - v \geq v - u$) of value accumulation. Instead, there are two "threshold" values, $u + \frac{1}{2}v$ and $\frac{3}{2}v$, which play important roles here. When x is between the thresholds, by the proof of Proposition 1, the singleton pool is strictly binded by the competition margin. When x is even lower than the low threshold, both pools are strictly binded by the competition margin. In contrast to these two scenarios, when x, with no upper bound, is larger than the high threshold, both pools are binded by the demand margin with no effect of the competition margin. These observations are consistent intuitively: the greater the value of complete pool (x) is with regard to those of incomplete pool and stand-alone patent, the less likely the competition margin is binding for both pools. Also, a pool of larger size is less likely to be binded by the competition margin *ceteris paribus*, since the marginal contribution to the complete pool of a bundle of patents is larger than that of a single patent.[14]

[14] In the standard model of output cartels in Cournot oligopoly, a small pool has higher per-owner profit

Proposition 2.

$$\pi\left(\{1,1,1\}\right) = \begin{cases} \left(\frac{1}{16\Delta}x^2\right)_3, & \text{if (d+f);} \\ \left(\frac{1}{\Delta}\left(3v-2x\right)\left(x-v\right)\right)_3, & \text{if (concavity+g);} \\ \frac{u-z}{\Delta}\left(z,z,x-u-z\right) \text{ with } z\in\left[v-u,\frac{x-u}{2}\right], & \text{if (e+convexity),} \end{cases}$$

where (d): $x > 2u$, (e): $x \le 2u$, (f): $x \ge \frac{4}{3}v$, and (g): $x < \frac{4}{3}v$.[15]

The dichotomy of concavity and convexity has effect on the categorization of profit profiles under the pool structure $\{1,1,1\}$. However, it is not the only decisive dichotomy. Furthermore, in the case (d+f), the accumulation of common value can be either concave or convex, and the competition margin does not work. On the contrary, by the proof of Proposition 2, all the three patents are strictly binded by the competition margin in the case (concavity+g). Notice that in the former case x has no upper bound given u and v, while x is doubly bounded from above in the latter case.[16] The tendency of being binded by the competition margin with low x under the fragmented structure resonates to our observation under the incomplete pool structure, as is explained above.

In the case (e+convexity)[17], $(z,z,x-u-z)$ is the equilibrium price profile charged by the owners. There exist infinite number of equilibria in this case; i.e., z, the price charged by the first two owners, can vary from $v-u$ to $\frac{x-u}{2}$ in equilibrium. Except when $z=\frac{x-u}{2}$, all of them are asymmetric equilibria in which singleton pool 3 makes the high profit while the other two make the same low profit. The smaller z is, the more asymmetric the equilibrium is. Thereby, z indicates the degree of symmetry (or fairness in a normative sense, in view of the *ex ante* symmetry of patents). Notice that the lower bound of z is the marginal value of a patent to an incomplete pool, and the upper bound is half of the marginal value of two patents to the complete pool. In other words, the owner in a disadvantageous position can guarantee his charge as what he contributes when forming an incomplete pool with another owner, and expect the highest price to be an equal split of the marginal value, when two owners together join the advantageous owner and create one complete pool.

than a big pool since every pool shares the same profit. However, this is not always the case in our model. Actually, it can be verified that $\{2\}$ earns higher per-owner profit than $\{1\}$ when (b) and $x < \frac{5}{4}v$ hold, or when (c) and concavity hold. This reflects the fact that, unlike the case where competition margin does not exist, the owner in a big pool may not be worse-off, since a small pool is hindered in its ability of pricing with the presence of competition margin (c.f., Proposition A1).

[15] For convenience, we use "(+)" to represent conditions holding jointly. These three groups of conditions, (d+f), (concavity+g), and (e+convexity) are disjoint and cover all the possible combinations of u, v and x. To see the completeness, notice that (concavity+f) implies (d), and (convexity+d) implies (f).

[16] The case (e+convexity) is a bit more complicated; see the proof of Proposition 2 for details. Generally speaking, owners are still vulnerable to the competition margin, especially when x is strictly bounded from above by (e).

[17] To make (e+convexity) hold, we require that $v \le \frac{3}{2}u$. If $v > \frac{3}{2}u$ and convexity hold, we fall into the case (d+f).

Furthermore, the profit of the advantageous (disadvantageous) owner decreases (increases) in z and reaches its minimum (maximum) in the polar case of symmetry.[18]

4 Stable pool structures

In this section, we restrict ourselves to the symmetric equilibrium under the pool structure $\{1,1,1\}$. Specifically, in the case (e+convexity), symmetry is obtained when $z = \frac{x-u}{2}$, and

$$\pi\left(\{1,1,1\}\right) = \left(\frac{1}{4\Delta}\left(3u-x\right)\left(x-u\right)\right)_3 .$$

Now we look at the details of stage 1 in game Γ: how the owners form the stable pool structure while having in mind the picture of profit profiles under different pool structures. Here we use the notion of *equilibrium binding agreements (EBA)*, suggested by Ray and Vohra (1997), as the protocol of pool formation in our analysis.[19] And we call a pool structure an *equilibrium pool structure (EPS)* if, under this structure, neither a single owner nor a group of owners *within* one pool has incentive to break away from the current pool by using the protocol of EBA. We aim to find the coarsest EPS as the *stable pool structure*. Two main features of EBA are as follows[20]:

(1) Only internal deviations of a subset of an existing pool are allowed, and there is no opportunity of re-merging after break-ups or cooperation among the owners in different pools. In this sense $\{1,1,1\}$ is an EPS, since every pool is singleton and there is no further chance of defection.

(2) When one owner considers breaking away from the complete pool, he does not assume that the resulting structure is $\{1,2\}$. Instead, he evaluates whether the complementary coalition $\{2\}$ will split even further and the pool structure end up in $\{1,1,1\}$. Similarly, when the coalition $\{2\}$ considers breaking away from the complete pool, both members have in mind the possibility of betrayal from his fellow member and $\{1,1,1\}$ as the resulting structure.[21] In this sense, the owners are farsighted in the course of coalition formation.

To illustrate the idea of EBA, consider the baseline model of output cartels in Cournot

[18] To see this, notice that $\frac{1}{\Delta}(u-z)(x-u-z)$ reaches its global minimum at $z = \frac{x}{2}$ and $\frac{x}{2} > \frac{x-u}{2}$; $\frac{1}{\Delta}(u-z)\,z$ reaches its global maximum at $z = \frac{u}{2}$ and $\frac{u}{2} \geq \frac{x-u}{2}$ by (e).

[19] This notion is further extended using von Neumann and Morgenstern (1944) abstract stable set by Diamantoudi and Xue (2007). For applications of EBA, see Ray and Vohra (1997) and Levy (2004). For an overview of different protocols of coalition formation, see, for example, Bloch (2003) and Yi (2003).

[20] Another important feature of EBA, named "the best response property", is incorporated in the specification of stage 2. In view of that, we say that the agreements within pools are binding, while there is no precommitment across pools.

[21] However, under the pool structure $\{1,2\}$, when one owner considers breaking away from $\{2\}$, the only candidate for the resulting structure is $\{1,1,1\}$.

oligopoly, as the case (a+d+f) in our setting:

$$\pi(\{3\}) = \frac{1}{12\Delta}x^2,\ \pi(\{1,2\}) = \left(\frac{1}{9\Delta}x^2, \frac{1}{18\Delta}x^2\right) \text{ and } \pi(\{1,1,1\}) = \left(\frac{1}{16\Delta}x^2\right)_3.$$

When one owner (*leading perpetrator*) considers defecting from the complete pool, he does not assume that the other two owners remain together. Instead, he takes into account their reactions to his defection. Since $\frac{1}{18\Delta}x^2 < \frac{1}{16\Delta}x^2$, once the leading perpetrator leaves the complete pool, the temporary coalition of the other two owners will break apart and the pool structure will be further changed to $\{1,1,1\}$. We say that $\{1,1,1\}$ *blocks* $\{1,2\}$. Therefore, the leading perpetrator will compare $\pi(\{3\})$ with $\pi(\{1,1,1\})$ instead of $\pi(\{1,2\})$. Since $\frac{1}{12\Delta}x^2 > \frac{1}{16\Delta}x^2$, no owner has incentive to part from his fellow members ($\{1,1,1\}$ does not block $\{3\}$), and then the complete pool, as the coarsest EPS, is the stable pool structure.

Based on EBA, we have the following simple algorithm to find the stable pool structure.

Step I. By comparing the per-owner profit of $\{2\}$ in $\pi(\{1,2\})$ with the profit of $\{1\}$ in $\pi(\{1,1,1\})$, answer the question: Is $\{1,2\}$ an EPS?[22] If YES, go to step III. Otherwise, go to step II.

Step II. By comparing the per-owner profit in $\pi(\{3\})$ with the profit of $\{1\}$ in $\pi(\{1,1,1\})$, answer the question: Is $\{3\}$ an EPS? If YES, $\{3\}$ is the coarsest EPS. Otherwise $\{1,1,1\}$ is the coarsest EPS. The algorithm stops.

Step III. Answer the same question as in step II, however, by checking whether $\{1\}$ *or* $\{2\}$ has incentive to defect from the complete pool. That is to say, compare the per-owner profit in $\pi(\{3\})$ with the profit of $\{1\}$ and the per-owner profit of $\{2\}$ in $\pi(\{1,2\})$ respectively. If YES, $\{3\}$ is the coarsest EPS. Otherwise $\{1,2\}$ is the coarsest EPS. The algorithm stops.

One useful observation is that $\{3\}$ is the stable pool structure when we end up in step II. This is because the complete pool maximizes the total profit of the three owners.[23] As an example of this algorithm, reconsider the case (a+d+f). In step I, by the analysis above, the answer is NO. Then we proceed to step II and know immediately that $\{3\}$ is the stable pool structure by the previous observation. Also, this implies that the fragmented pool structure can never be stable in the symmetric scenario.

Since the cases (c+d+f) and (a+concavity+g) do not exist[24], altogether, there are seven cases we need to investigate in terms of different combinations of conditions (a)-(g) and concavity/convexity. In addition to the case (a+d+f), the other six cases with more complexities are analyzed below using the algorithm.[25]

[22] A rephrasing of the question is that: Does $\{1,1,1\}$ block $\{1,2\}$? If YES, then $\{1,2\}$ is NOT an EPS.

[23] See Proposition 7.1(ii) in Ray and Vohra (1997).

[24] To see the former, notice that (c+d) implies $v > 2u$, and (c+f) implies $v < \frac{6}{5}u$.

[25] For the sake of brevity of our presentation, we focus on the full-fledged subcases where all the critical

4.1 (d+f)

When (b+d+f)[26], the equilibrium under the pool structure $\{1,1,1\}$ is the same as in the baseline case (a+d+f). However, the change of equilibrium under the pool structure $\{1,2\}$ leads to a big difference of the stable pool structure. In particular, there are two subcases in terms of a cut-off value $\sqrt{2}v$. When $x > \sqrt{2}v$, similar to the baseline case (a+d+f), $\{1,2\}$ is not an EPS (step II) for $\frac{1}{8\Delta}v^2 < \frac{1}{16\Delta}x^2$ and $\{3\}$ is the coarsest EPS. On the contrary, when $x \leq \sqrt{2}v$, $\{1,2\}$ becomes an EPS (step III) and algebra in the proof of Proposition 3 shows that $\frac{1}{12\Delta}x^2 < \frac{1}{2\Delta}v\,(x-v)$. Therefore, $\{1\}$ has incentive to break away from the complete pool[27] and the coarsest EPS is $\{1,2\}$.

Proposition 3. *Let $x_1 \equiv \sqrt{2}v$. In the case (b+d+f), when $x > x_1$, the stable pool structure is $\{3\}$; when $x \leq x_1$, the stable pool structure is $\{1,2\}$.*

4.2 (concavity+g)

Proposition 4. *Let $x_2 \equiv \sqrt{\frac{3}{2}}v$, $x_3 \equiv \left(3-\sqrt{3}\right)v$, and $x_4\,(x_5) \equiv \frac{1}{7}\left(6u+3v-(+)\sqrt{3\delta}\right)$, where $\delta \equiv 3v^2-2uv-2u^2$ if $\delta \geq 0$. In the case (b+concavity+g), when $x_2 \leq x \leq x_3$, the stable pool structure is $\{3\}$; otherwise, the stable pool structure is $\{1,2\}$. In the case (c+concavity+g), when $x_4 < x < x_5$, the stable pool structure is $\{1,2\}$; otherwise, the stable pool structure is $\{3\}$.*[28]

In both cases, the pool structure $\{1,2\}$ becomes an EPS. Furthermore, there appears a plethora of outcomes caused by different leading perpetrators. In the case (b+concavity+g), when $x < x_2$, $\{2\}$ is the leading perpetrator from the complete pool, while $\{1\}$ breaks away when $x > x_3$. When $x_2 \leq x \leq x_3$, neither $\{2\}$ nor $\{1\}$ induces a defection. In the case (c+concavity+g), $\{2\}$ is the leading perpetrator when $x_4 < x < x_5$, while a single owner never defects. It is noted that, because of the quadratic profit function, the stable pool structure does not change monotonically when x varies in either case.

values of x, listed in Propositions 3-5, are attainable. All the relevant conditions for attainability are given in the footnotes or the proofs. So are the conditions which make these six cases occur.

[26] To make the set of x's satisfying (b+d+f) nonempty, we must have $2u < \frac{3}{2}v$ and hence $v > \frac{4}{3}u$. This, in turn, implies that $u + \frac{1}{2}v < \frac{4}{3}v$. Hence (b) and (f) are equivalent to $\frac{4}{3}v \leq x < \frac{3}{2}v$. Notice that in Proposition 3, $x_1 \equiv \sqrt{2}v \in \left(\frac{4}{3}v, \frac{3}{2}v\right)$.

[27] This implies that $\{2\}$ has no incentive to break away from the complete pool, otherwise the total profit in the pool structure $\{1,2\}$ would be higher than that in $\{3\}$.

[28] To make the set of x's satisfying (b+concavity+g) nonempty, we must have $v > \frac{4}{3}u$. In addition, when $v > \frac{1}{\sqrt{3/2}-1/2}u \left(> \frac{4}{3}u\right)$, both x_2 and x_3 satisfy (b+concavity+g). Without the latter condition, however, our result does not change. Also, x_4 always satisfies (c+concavity+g), and so does x_5 if $v < \frac{2\sqrt{6}+2}{5}u$ and $v \neq \frac{5}{4}u$. Still, whether x_5 satisfies (c+concavity+g) does not affect our result.

4.3 (e+convexity)

Proposition 5. *Let* $x_6 \equiv 2u - \frac{\sqrt{2}}{2}\sqrt{2u^2 - v^2}$*, if* $2u^2 - v^2 > 0$*. In the cases (a/c+e+convexity), the stable pool structure is* $\{3\}$*. In the case (b+e+convexity), when* $x_3 < x \leq x_6$*, the stable pool structure is* $\{1, 2\}$*; otherwise, the stable pool structure is* $\{3\}$*.*

In the cases (a/c+e+convexity), the pool structure $\{1, 2\}$ is not an EPS and we end up in the complete pool. Unlike all the other cases, in the case (b+e+convexity), whether $\{1, 2\}$ is an EPS is indeterminate, and depends on the magnitude of x in terms of a cut-off value x_6. When it is ($x \leq x_6$), $\{2\}$ is never the leading perpetrator breaking away from the complete pool, while a single owner benefits from defection when $x > x_3$.

4.4 Summary and examples

The following table summarizes the results of Propositions 3-5 and some pivotal information about pool formation under different pool structures.

When is the coarsest EPS	$\{3\}$?	$\{1, 2\}$?	Is $\{1, 2\}$ an EPS?	Who defects?*
a+d+f	Always		Never	
b+d+f	$x > x_1$	$x \leq x_1$	$x \leq x_1$	$\{1\}$
b+concavity+g	$x \in [x_2, x_3]$	$x \notin [x_2, x_3]$	Always	$\{2\}$ [/] $\{1\}$**
c+concavity+g	$x \notin (x_4, x_5)$	$x \in (x_4, x_5)$	Always	/ ($\{2\}$) /
a/c+e+convexity	Always		Never	
b+e+convexity	$x \notin (x_3, x_6]$	$x \in (x_3, x_6]$	$x \leq x_6$	/ ($\{1\}$]

(*The complete question is "Who defects from $\{3\}$ if $\{1, 2\}$ is the coarsest EPS?" **The notation means that when $\{1, 2\}$ is an EPS, $\{2\}$ defects from $\{3\}$ if $x < x_2$ and $\{1\}$ defects if $x > x_3$. Other notations are used similarly.)

Based on what we find in Propositions 3-5, some remarks are made as follows. First, when $\{1, 2\}$ is the stable pool structure, in the cases with concavity, a two-owner group is normally the leading perpetrator (except the subcase with $x > x_3$), while in the cases with convexity, a single owner is the only possible leading perpetrator. Intuitively, this corresponds to the fact that with a concave value function, an incomplete pool constitutes the most part of complete pool's value, while with a convex value function, a single patent contributes greatly to the complete pool.

Second, the following proposition shows that, as long as x is sufficiently large compared with v, or v is sufficiently small compared with u, the stable pool structure is (almost) the complete pool. In other words, when the complete pool is valued highly, or the incomplete pool is valued poorly, the owners tend to end up in capturing the highest total profit by forming a grand coalition. Surprisingly, the latter condition is irrespective of x.

Proposition 6. *If $x > x_1$, except the case (b+e+convexity), the stable pool structure is $\{3\}$; furthermore, if (a), the stable pool structure is always $\{3\}$. If $v < \frac{4}{3}u$, except the case (c+concavity+g), the stable pool structure is $\{3\}$; furthermore, if $v < \frac{\sqrt{7}+1}{3}u$, the stable pool structure is always $\{3\}$.*

Last, to further illustrate the implications of Propositions 3-5, we provide some examples of interest with specific value functions. (In all the following examples, ϵ is an infinitesimal number.)

Example 1. Consider one technology to which one patent is sufficient, and more patents added are redundant. That is, let $v = u + \epsilon$ and $x = u + 2\epsilon$. By Proposition 6, we know immediately that $\{3\}$ is the stable pool structure. More specifically, this is the case (c+e+convexity).

Example 2. Consider one technology which performs at its maximum with two patents. Meanwhile, individual use of patent generates no value. That is, let $u = \epsilon$ and $x = v + \epsilon$. It can be verified that this value function satisfies (b+concavity+g). Furthermore, $x < x_2$, and by Proposition 4, $\{1, 2\}$ is the stable pool structure. Notice that $\{2\}$, with exactly the critical number of patents the technology requires, has incentive to break away from the complete pool.

Example 3. Consider one technology consisting of all three patents, which have no values if used in any other way. That is, let $u = \epsilon$ and $v = 2\epsilon$. By Proposition 6, the stable pool structure is $\{3\}$. More specifically, it falls into the baseline case (a+d+f).

Example 4. Consider one technology with a linear value function. That is, let $V(k) = Ak - \overline{\theta}$ $(k = 1, 2, 3)$, where A is a positive coefficient of value accumulation. It can be verified that this linear value function satisfies (b+d+f), and furthermore, $x > x_1$. Therefore, by Proposition 3, the stable pool structure is $\{3\}$.

Example 5. Consider one technology with a power value function in the form of $V(k) = k^\alpha - \overline{\theta}$ $(k = 1, 2, 3)$, where $\alpha > 0$, $\alpha \neq 1$. When the value function is convex $(\alpha > 1)$, (a+d+f) is the case with the complete pool as the stable pool structure. When the value function is concave $(\alpha < 1)$, varieties of the stable pool structure arise. For example, if $\alpha \in \left(\log_{3/2}\left(3 - \sqrt{3}\right), \log_{3/2}\sqrt{2}\right)$, the stable pool structure is $\{1, 2\}$ by Propositions 3 and 4. [More specifically, when $\alpha \geq \frac{2\ln 2 - \ln 3}{\ln 3 - \ln 2} \in \left(\log_{3/2}\left(3 - \sqrt{3}\right), \log_{3/2}\sqrt{2}\right)$, it is the case (b+d+f); otherwise value function satisfies (b) and (g).] If $\alpha \in \left(\frac{1}{2}, \log_{3/2}\left(3 - \sqrt{3}\right)\right]$, it falls into the case (b+concavity+g), and, by Proposition 4, the stable pool structure is $\{3\}$ since $x_2 \leq x \leq x_3$.

4.5 Welfare analysis

Based on the analysis of the stable pool structure above, we can investigate its effect on consumer surplus. We say that the stable pool structure *increases welfare* if the total price under the stable pool structure is lower than that under the fragmented structure. The following proposition shows that among all the cases, four cases always lead to welfare improvement, and so do the other cases under some circumstances.[29]

Proposition 7. *In the cases (a/b+d+f), (a/b+e+convexity), the stable pool structure always increases welfare. In the case (b+concavity+g), the stable pool structure increases welfare if and only if $x \geq x_2$. In the case (c+e+convexity), the stable pool structure increases welfare if and only if $x > \frac{3}{2}u$. In the case (c+concavity+g), the stable pool structure always decreases welfare except when $v > \frac{5}{4}u$ and $x \geq x_5$ hold together.*

Proposition 7 shows that, except the case (c+concavity+g), which has a very restrictive value of complete pool, in all the other cases, the stable pool structure has a tendency of increasing the consumer welfare (as long as the value of complete pool is large enough in some cases). For instance, in the benchmark case (a+d+f), the stable complete pool, when internalizing the negative strategic externalities across individual owners, decreases the total price from $\frac{3}{4}x$ to $\frac{1}{2}x$.[30]

5 Asymmetric games

As we have seen in Proposition 2, though the patents are symmetric *ex ante*, there exists asymmetric equilibria under the fragmented pool structure. Particularly, one owner (called A) may have a pricing advantage and earn higher profit than the other two (called a). To investigate some properties of the stable pool structure when allowing for asymmetric equilibria, we need to distinguish between two different pool structures with one incomplete

[29]To break the tie, we ignore the boundary situations where two pool structures share the same total price.

[30]It can also be shown that in the cases (a/b+d+f), (a/b+e+convexity) and (b+concavity+g) except when $x < \frac{6}{5}v$, the complete pool charges the lowest total price. These are the cases where at least the incomplete pool is binded by the demand margin, and this observation is consistent with the behaviors we usually expect from the players with strategic substitutes. In all the cases with (c), as long as x is not too large ($x > \frac{6}{5}v$ in the case with concavity, $x > \frac{3}{2}u$ in the case with convexity), the fragmented structure always leads to the lowest total price. This reflects the fact that being binded by the competition margin hinders the owner's ability of pricing.

pool: $\{a, aA\}$ with one a and A forming one pool, and $\{A, aa\}$ with stand-alone A. Similarly, we denote the complete pool by $\{aaA\}$, and the fragmented structure $\{a, a, A\}$.

The algorithm of finding the stable pool structure, based on the notion of EBA, needs to be modified accordingly. Generally speaking, we are supposed to check whether $\{a, aA\}$ and $\{A, aa\}$ are EPS respectively, and then, based on that, evaluate the stability of $\{aaA\}$. Subtleties of the algorithm arise here. When both $\{A, aa\}$ and $\{a, aA\}$ are EPS, it is sufficient to check whether $\{aaA\}$ is blocked by either of the incomplete pool structures. On the other hand, when only one of the incomplete pool structures is an EPS, by the definition of EBA, we need to check whether the complete pool is blocked by this EPS and $\{a, a, A\}$ which is always an EPS. However, in our setting, when $\{A, aa\}$ is an EPS and $\{a, aA\}$ is not, we only need to check whether $\{A, aa\}$ blocks $\{aaA\}$. The reason is as follows. Since $\{A, aa\}$ is an EPS, $\{a, aA\}$ would be on the blocking route if $\{a, a, A\}$ blocked the complete pool. This implies that $\{a\}$ or $\{aA\}$ is the leading perpetrator. However, recall that even in the symmetric game, a single (farsighted) owner in the complete pool who earns higher profit than a has no incentive to end up in the fragmented structure, and thus, there is no possibility of $\{a, a, A\}$ blocking $\{aaA\}$.[31]

One asymmetric game of special interest is that when A earns the highest profit, $z = v - u$, and

$$\pi\left(\{a, a, A\}\right) = \frac{2u - v}{\Delta}\left(v - u, v - u, x - v\right).$$

Instead of fully characterizing the stable pool structure, which is a repetitive exercise as in Section 4, the following proposition demonstrates that the coarsest EPS tends to be finer in the most asymmetric scenario. (Recall that in the cases (a/c+e+convexity) with symmetric equilibrium, the complete pool is always stable.) In the proof of Proposition 8, it is shown that $\{A, aa\}$ becomes an EPS deterministically in all the cases with (e+convexity).[32] Intuitively, considering the worst-off situation under the fragmented structure, the two disadvantageous owners are willing to stick together for the sake of higher profits, and thus stabilize the incomplete pool structure.

Proposition 8. *When $z = v - u$, $\{A, aa\}$ can be the stable pool structure in all the cases with (e+convexity), and $\{a, a, A\}$ is never the stable pool structure in any case.*

Let $\pi\left(a|A, aa\right)$ denote the profit of a under the pool structure $\{A, aa\}$, and other notations are defined in a similar way. In view of Proposition 8, one may wonder whether the

[31] This reasoning is covered by the definition of EBA, particularly (B.3) in Ray and Vohra (1997). The fact that A is the only possible leading perpetrator from $\{aaA\}$ to $\{a, a, A\}$ implies that re-merging of the other owners *except* the leading perpetrator leads to a pool structure which is also blocked by $\{a, a, A\}$. Obviously, this is not the case, since $\{A, aa\}$ is an EPS.

[32] By contrast, $\{a, aA\}$ must not be an EPS whenever $\{1, 2\}$ is not an EPS in the symmetric counterpart.

fragmented structure $\{a, a, A\}$ would be the stable pool structure in some case.[33] This can only occur when none of the pool structures, except $\{a, a, A\}$ itself, are EPS. Formally, this requires that

$$\begin{aligned} \pi(a|A, aa) &< \pi(a|a, a, A), \qquad (5) \\ \text{and } \pi(A|aaA) &< \pi(A|a, a, A). \qquad (6) \end{aligned}$$

Notice that (5) also implies that $\{a, a, A\}$ blocks $\{a, aA\}$ with A defecting.[34] The following proposition gives us a possibility answer in the case with (a). Interestingly, we illustrate that the fragmented structure as the stable pool structure occurs when the degree of symmetry, z, is in the middle of its range, other than either of the polar districts. That is to say, in our setting with the formation protocol of EBA, the finest market structure outcome results from the modest asymmetry (unfairness) instead of any extreme situations. The intuition behind that is clear. To make the fragmented structure prevail, both (5) and (6) should hold simultaneously. Nevertheless, the former condition is violated under the extreme asymmetry (as is shown in the proof of Proposition 8), while the latter never works in the symmetric scenario. Correspondingly, when the degree of symmetry is sufficiently high (low), the stable pool structure is consistent with what we observe in the symmetric (extremely asymmetric) game, as is stated in Proposition 5 (8).

Proposition 9. *In the case (a+e+convexity), there exists* $(\underline{z}, \overline{z}) \subset \left[v-u, \frac{x-u}{2}\right]$ *such that the stable pool structure is* $\{A, aa\}$ *when* $z \in [v-u, \underline{z}]$*,* $\{a, a, A\}$ *when* $z \in (\underline{z}, \overline{z})$*, and* $\{aaA\}$ *when* $z \in \left[\overline{z}, \frac{x-u}{2}\right]$*.*[35]

There is some welfare implication when allowing for asymmetric equilibria. In the case (a+e+convexity) with symmetric equilibrium, as is shown in Proposition 5, the complete pool is always the stable pool structure, and hence the consumer welfare is represented by the price charged by the complete pool, $\frac{1}{2}x$. This is also the case when the degree of symmetry is high enough $\left(z \in \left[\overline{z}, \frac{x-u}{2}\right]\right)$ by Proposition 9. However, when the equilibria are sufficiently asymmetric ($z \in [v-u, \underline{z}]$), the consumer welfare is reduced with the incomplete pool structure charging the total price of $\frac{2}{3}x$. When the fragmented pool structure is stable ($z \in (\underline{z}, \overline{z})$), the total price is $x-u+z$, increasing in z, and welfare effect is contingent on the value of complete pool. As is summerized in the following proposition, the total price under the stable fragmented pool structure may increase or decrease compared with that under the complete pool, and the higher x is, the lower the possibility is of improving the

[33]Furthermore, we can show a strong impossibility result: In the case (c+e+convexity), $\{a, a, A\}$ is never the stable pool structure for *any* z.

[34]This is because $\pi(A|a, aA) = \pi(a|A, aa)$ and $\pi(a|a, a, A) < \pi(A|a, a, A)$.

[35]The values of $\underline{z}$ and $\overline{z}$ are given in the proof.

consumer welfare.

Proposition 10. *In the case (a+e+convexity), when* $x < \frac{18}{11}u$, *there exists* $z^* \in (\underline{z}, \overline{z})$ *such that* $z \in (\underline{z}, z^*)$ *[*$z \in (z^*, \overline{z})$*] leads to a lower [higher] total price than the one charged by the complete pool; when* $x \geq \frac{18}{11}u$, *any* $z \in (\underline{z}, \overline{z})$ *leads to a higher total price.*[36]

6 Discussion

6.1 Sequential bargaining as an alternative protocol

Another prevailing protocol of coalition formation in the literature, in the spirit of sequential bargaining, is the *infinite-horizon unanimity game*, suggested by Bloch (1996) and generalized by Ray and Vohra (1999).[37] According to some order of moves, the first owner proposes a pool including himself to a group of owners (or leaves the game as a singleton pool). If the proposal is accepted sequentially by all the members of this pool, this pool leaves the game and the remaining owners continue playing the game. If one potential member rejects the current proposal, the game shifts to a new start initiated by a proposal from this rejector. When every owner belongs to a pool, the unanimity game stops and we proceed to the second stage of the game Γ. If the unanimity game continues forever, all the owners get payoffs of zero. The main results in Bloch (1996) and Ray and Vohra (1999) show that in the symmetric game a focal prediction of the (stationary perfect) equilibrium coalition structure can be derived by finding a subgame perfect equilibrium of a sequential game of choosing pool size, which proceeds as follows. The first owner chooses an integer $k_1 \in [1, n]$ and the first k_1 owners form a pool. Then the $(k_1 + 1)$-th owner chooses an integer $k_2 \in [1, n - k_1]$ as the size of the next pool and the game continues to the move of the $(k_1 + k_2 + 1)$-th owner. The game stops when all the owners are exhausted.

It is straightforward to see that in our three-patent symmetric game, the coarsest EPS coincides with the coalition structure resulting from the subgame perfect equilibrium of the game of choosing pool size. Therefore, these two protocols of pool formation give us a common solution to the stable pool structure.[38]

[36] This proposition is of special interest if z can be adjusted to improve the consumer welfare. When $x < \frac{18}{11}u$, the highest welfare arrives at z slightly higher than $\underline{z}$. Otherwise, the (almost) symmetry is the best choice.

[37] See Bloch (1995), Yi (1998), Morasch (2000), Ray and Vohra (2001) and Greenlee (2005) for its applications. Ray's (2007) seminal book on coalition formation is devoted entirely to these two protocols, the blocking approach we use in Section 4 and the bargaining approach. The former is more in the spirit of cooperative game theory, while the latter of noncooperative bargaining.

[38] This invariance also follows a general observation: In a symmetric three-player partition function game with generic payoffs, the coarsest equilibrium coalition structure based on EBA is equivalent to the coalition structure resulting from the subgame perfect equilibrium of the game of choice of coalition sizes.

6.2 An n-patent case with homogenous licensees

In this subsection, we provide a preliminary discussion on the complexity of endogenous pool formation in the general case with n intellectual property owners. In the preceding part of the paper, the licensees are assumed to be heterogeneous with a variable of private value θ following some distribution. Naturally, a simpler case is that with a continuum of licensees with the same private value denoted by θ. Still, assume that $\theta + V(n) > 0$ and θ is sufficiently large to guarantee the adoption of technology. In particular, in stage 4, the licensees adopt the technology if and only if $\theta + V(\sharp B) \geq \mathbf{P}_B$. The rest part of the model is the same. Similarly, given a pool structure $C = \{n_1, n_2, ..., n_m\}$ formed in stage 1, we have a subgame $\Gamma(C)$ comprising stages 2-4. All the pools are in the equilibrium basket B. More importantly, in equilibrium only competition margins work, and the equilibrium price profile $\boldsymbol{p} \equiv (p_1, p_2, ...p_m)$, which may not be unique, satisfies the following system of equations

$$p_i = V(n) - \mathbf{P}_{-i} - \max_{J \subseteq C \backslash n_i} \{V(\sharp J) - \mathbf{P}_J\}, \text{ for every } n_i \in C.$$

Obviously, in a three-patent case, the coarsest EPS is unique since any two of pool structures are comparable in terms of coarseness. On the contrary, in a n-patent case even with homogenous licensees and no externalities, there may exist multiple stable pool structures, as is shown in the following example.

To further remove the externalities across the pools, we assume that either condition in Proposition A2 (see Appendix A) is satisfied. Let $w(t) \equiv V(n) - V(n-t)$ for $t \leq n$, and then $w(t)/t$ is the average marginal contribution to the complete pool of a pool of size t. By Proposition A2, given a pool structure $C = \{t_1, t_2, ..., t_m\}$, the profit profile $\pi(C)$ in the last stage is

$$\left(\frac{w(t_1)}{t_1}, \frac{w(t_2)}{t_2}, ..., \frac{w(t_m)}{t_m}\right).$$

Notice that $\frac{w(t_i)}{t_i}$ is irrespective of the ambient pool structure.[39]

Example 7. Let $n = 6$. Consider the following value function $V(t)$ and its corresponding per-owner profit $\pi(t)$

t	1	2	3	4	5	6
$V(t)$	11.5	14	36	46	53	54
$w(t)/t$	1	4	6	10	8.5	9

By the notion of EBA, there are two coarsest equilibrium (and hence, stable) pool structures, $\{2, 4\}$ and $\{3, 3\}$, both consisting of two pools. (Note that $\{1, 5\}$ is blocked by the EPS

[39] Here the partition function is reduced to a characteristic function, which is the more conventional analysis tool in cooperative game theory, $v : \{t | t \in \mathbf{N}, t \leq n\} \to \mathbb{R}^t$ such that $v(t) = \left(\frac{w(t)}{t}\right)_t$.

$\{1, 1, 4\}$, and the complete pool is blocked by $\{2, 4\}$.) Also, it shows that when the number of patents is larger than 3, any stable pool structure may consist of multiple incomplete pools without any stand-alone patent.

This example indicates that if we want a sharper prediction of the stable coalition structure, the notion of EBA needs to be refined. One way to do this is to require that the coarsest EPS block some coarser pool structure. In Example 7, $\{3, 3\}$ does not satisfy this requirement. Another way is to use the bargaining approach with specific negotiation process, which we discuss above. With generic payoffs, it will lead to a unique prediction of pool structure in subgame perfect equilibrium, which is a concatenation of pools $\{t^1, t^2, ...t^m\}$ satisfying

$$t^1 = \arg\max_{t \leq n} \frac{w(t)}{t},\ t^j = \arg\max_{t \leq n - \sum_{k<j} t^k} \frac{w(t)}{t} \text{ for } j > 1, \text{ and } \sum_{j=1}^{m} t^j = n.$$

It also provides the prediction of $\{4, 2\}$ for Example 7.

7 Conclusion

In this paper, we study the endogenous coalitional behaviors of intellectual property owners in a three-patent setting. Based on a general characterization of equilibrium in the n-patent case, we fully characterize the equilibria under different pool structure with three patents. A striking interaction between the demand margin and the competition margin gives rise to a plethora of equilibrium scenarios. In particular, the competition margin works more often under any pool structure with a low-value complete pool, and it affects less a pool of large size. Also, it is noted that under the fragmented pool structure, the number of equilibria can be infinite, and the class of asymmetric equilibria drives some interesting results on stable pool structures and welfare effects thereof.

In terms of different combinations of decisive conditions, including concavity and convexity, there are seven cases in which the endogenous pool formation is investigated in a symmetric game. When addressing the stability of different pool structures, we resort to the notion of equilibrium binding agreement. This notion incorporates the farsightedness and internal deviation of owners, and is well suited to our three-patent game. We implement a simple algorithm to check which feasible pool structure is the stable one, and the identification of incomplete pool structure becomes a pivotal link in our analysis. Not surprisingly, there is no straightforward prediction of the stable pool structure, but the fragmented pool structure is never stable. Moreover, the complete pool always forms, if its value is sufficiently large or the value of incomplete pool is sufficiently small. Often are consumers better-off under the stable pool structure, as long as the complete pool is highly valued. These results are robust in the sense that nothing is changed when a sequential unanimity game is adopted as the pool formation protocol instead of equilibrium binding agreement.

An interesting variant is the one introducing asymmetric profit equilibria. In the most asymmetric scenario, though the fragmented structure is still impossible to be formed, the stable pool structure tends to be finer than the symmetric counterpart. More importantly, the stability of the fragmented pool structure is eventually established in the range of moderate asymmetry in one subcase. In addition, its effect on the consumer welfare may be positive.

To conclude our paper, we point out some potential directions for further research. First, in our analysis patent owners collect up-front fees from licensees who access their patents. Except up-front fees, per-unit royalties and combinations of the two are used in practice (Taylor et al. 1973). It may be interesting to study the effect of different licensing policies on endogenous pool formation, and sharpen our understanding of issues related to patent licensing.[40] Second, in our paper the terms of intellectual property (owner) and patent are used interchangeably. However, in reality uncertain situations arise related to licensing and use of intellectual properties, for example, patent litigation (Choi 2010) and spillovers across intellectual properties (Yi and Shin 2000). An enriched model is necessary to study the endogenous coalition formation of intellectual properties when patents are "weak". Last, an extension of the current model to a general n-patent case is far more than trivial.

Appendix A

First, we characterize and show the existence of the subgame perfect equilibrium of $\Gamma(C)$ introduced in Section 2.

Proposition A1 (Lerner and Tirole 2004). *Given $C \equiv \{n_1, n_2, ..., n_m\}$ with $n_1 \leq n_2 \leq ... \leq n_m$, there exists (at least) one equilibrium of $\Gamma(C)$. Any equilibrium basket is C. Any equilibrium price profile is in the form of $\boldsymbol{p} \equiv (z_1, z_2, ...z_{m'}, \widehat{p}, ..., \widehat{p})$ satisfying the following conditions:*

(a) $0 \leq m' \leq m$ ($\boldsymbol{p} \equiv (\widehat{p})_m$ when $m' = 0$, $\boldsymbol{p} \equiv (z_i)_{i=1}^m$ when $m' = m$);
(b) $z_i = z(\boldsymbol{p}_{-i})$ for $i = 1, ..., m$;
(c) If $m' < m$, $\widehat{p}$ is defined by $\widehat{p}D'(\mathbf{P}) + D(\mathbf{P}) = 0$. And $\widehat{p} > z_j$ for $j = 1, ..., m'$; $\widehat{p} \leq z_j$ for $j = m' + 1, ..., m$;

[40] There is abundant literature on game-theoretic models of patent licensing. See Kamien (1992) for a survey, and Sen and Tauman (2007) and references thereof for recent study.

(d) If $m' = m$, $z_i < r(\boldsymbol{p}_{-i})$ for $i = 1, ..., m$.

Proof: (Existence) Let $p_i \in X \equiv \left[0, \overline{\theta} + V(n)\right]$, the nonempty compact convex set of each patent pool's price candidates. Patent pool i's best response function is $BR(\boldsymbol{p}_{-i}) = \min\left\{z(\boldsymbol{p}_{-i}), r(\boldsymbol{p}_{-i})\right\}$. Obviously, $r(\boldsymbol{p}_{-i})$ and $z(\boldsymbol{p}_{-i})$ are continuous. Hence the function $\left(BR(\boldsymbol{p}_{-i})\right)_{i=1}^{m} : X^m \to X^m$ is continuous. By Brouwer's fixed point theorem, there exists a fixed point $\boldsymbol{p}^*$ which is in the equilibrium of $\Gamma(C)$.

(Equilibrium basket) Assume by absurdity that there exists an equilibrium price profile $\boldsymbol{p} \equiv \left(\boldsymbol{p}_{-i}, p_i\right)$ with $n_i \notin B$ for some patent pool i. Let p_i' be a (positive) price lower than $V(\sharp B + n_i) - V(\sharp B)$, then $V(\sharp B + n_i) - \mathbf{P}_B - p_i' > V(\sharp B) - \mathbf{P}_B \geq V(\sharp B') - \mathbf{P}_{B'}$ for any B' with $n_i \notin B'$. The second inequality comes from the fact that B is the maximizer of (1) under $\boldsymbol{p}$. Thus, under $\left(\boldsymbol{p}_{-i}, p_i'\right)$, n_i belongs to the "new" basket. Hence, the patent pool i has incentive to lower its price from p_i to p_i', in view of earning positive profit.

(Equilibrium prices) Proposition 6(i) in Lerner and Tirole (2004) shows that the prices satisfying conditions (a)-(d) are in the equilibrium. Here we show the opposite. Suppose that $p_i = \widehat{p}(\leq z_i)$ for some i. To show that for any $j > i$, $p_j = \widehat{p}$, assume that $p_j \neq \widehat{p}$ by absurdity. (The necessity of the rest part of conditions (a)-(d) is apparent.) Then $p_j = z_j < \widehat{p}$. By definition, $z_i - z_j = p_i - p_j - M$, where $M \equiv \max_{K \subseteq C \backslash n_i} \left\{V(\sharp K) - \mathbf{P}_K\right\} - \max_{K' \subseteq C \backslash n_j} \left\{V(\sharp K') - \mathbf{P}_{K'}\right\}$. Hence $z_i = \widehat{p} - M$. However, since $n_j \geq n_i$ and $p_j < \widehat{p}$, M must be positive by the fact that $V(k)$ strictly increases in k. Therefore $z_i < \widehat{p}$. $\square$

Next, we discuss a special class of games of interest with competition margin $z_i = w_i$ in equilibrium for every pool i. Let $w(k, \Delta k) \equiv V(k) - V(k - \Delta k)$ for $k > \Delta k > 0$; i.e., $w(k, \Delta k)$ is the marginal contribution of Δk patents to a pool of size k (after the size expands). Denote by $w_i \equiv w(n, n_i)$ the marginal value of pool i to the complete pool. It is easy to see that w_i increases in n_i. The following proposition provides a characterization in terms of this class of games.

Proposition A2. *As described by Proposition A1, let* $\mathbf{p}$ *be an equilibrium price profile of* $\Gamma(C)$ *under some pool structure* C*. Then* $z_i = w_i$ *for* $i = 1, ..., m$ *if and only if* $\mathbf{p}$ *satisfies*

$$C \backslash n_i \in \arg\max_{J \subseteq C \backslash n_i} \left\{V(\sharp J) - \mathbf{P}_J\right\} \text{ for } i = 1, ..., m. \tag{7}$$

This condition holds for all the C*'s with* $\sharp C = n$*, if* $V(\cdot)$ *satisfies*

$$w(n, \Delta k) \leq w(k, \Delta k) \text{ for any } k \leq n \text{ and any } \Delta k < k. \tag{8}$$

Proof: By definition,

$$z_i = V(n) - \mathbf{P}_{-i} - \max_{J \subseteq C \backslash n_i} \{V(\sharp J) - \mathbf{P}_J\} \leq V(n) - \mathbf{P}_{-i} - \{V(n - n_i) - \mathbf{P}_{-i}\} = w_i.$$

The equality holds when $\max_{J \subseteq C \backslash n_i} \{V(\sharp J) - \mathbf{P}_J\} = V(n - n_i) - \mathbf{P}_{-i}$.

For every C with $\sharp C = n$, condition (7) is equivalent to that

$$V(n - n_i) - \mathbf{P}_{-i} \geq V(\sharp J) - \mathbf{P}_J \text{ for any } J \subseteq C \backslash n_i \text{ and any } n_i \in C.$$

Fix C, n_i and $J \subseteq C \backslash n_i$. Let $C \backslash n_i \backslash J \equiv \{n_{k_1}, n_{k_2}, ..., n_{k_m}\}$. If condition (8) holds, we have

$$\begin{aligned} V(n - n_i) - V(\sharp J) &= w(\sharp J + n_{k_1}, n_{k_1}) + w(\sharp J + n_{k_1} + n_{k_2}, n_{k_2}) + ... + w(n - n_i, n_{k_m}) \\ &\geq \sum_{n_j \in C \backslash n_i \backslash J} w_j \geq \mathbf{P}_{-i} - \mathbf{P}_J. \end{aligned}$$

The first inequality comes from condition (8), and the second the fact that $z_i \leq w_i$ for every i, and $\widehat{p} \leq z_j$ for $j = m' + 1, ..., m$ [condition (c) in Proposition A1]. $\square$

Condition (7) is a sufficient and necessary condition for $z_i = w_i$ in equilibrium for every pool i. A drawback seems to be that this condition is in terms of not only $V(\cdot)$ but also the equilibrium *per se*. To ensure that it can be satisfied under some circumstances, a sufficient condition (8), which is only based on $V(\cdot)$, follows. Condition (8) is weaker than, say, the condition of concavity of $V(\cdot)$ (Lerner and Tirole 2004). It does not require that $V(\cdot)$ have nonincreasing differences across all the patents, but that the terminal difference be the smallest. For example, consider a function $V(\cdot)$ with $V(1) = 1$, $V(2) = 2$, $V(3) = 5$ and $V(4) = 6$. Obviously, $V(\cdot)$ is not concave, while it satisfies condition (8): $w(4, \Delta k) \leq w(k, \Delta k)$ for any $\Delta k < k \leq 4$. Hence every $\Gamma(C)$ with $\sharp C = 4$ has an equilibrium in which w_i prevails as competition margins for every pool i.

Appendix B

Proof of Proposition 1: (We suppress $r(\boldsymbol{p}_{-i})$ and $z(\boldsymbol{p}_{-i})$ as r_i and z_i.)

Step 1. First notice that there are three possible cases in terms of the number m' of the pools *strictly* binded by the competition margin [case $m.m'$) with $m = 2$ and $m' = 0$, 1 or 2].

Case 2.0) Let the equilibrium price profile be $(\widehat{p})_2$. Then by equation (2), $\widehat{p} = \frac{x}{3}$. Also, by equation (4), $z_1 = x - \widehat{p} - (v - \widehat{p}) = x - v$, and $z_2 = x - \widehat{p} - (u - \widehat{p}) = x - u$.

Case 2.1) Let the equilibrium price profile be $(z_1, \widehat{p})$ with $z_1 < \widehat{p}$. (If pool 1 is weakly binded by the competition margin ($r_1 = z_1$), then by Proposition A1, $z_1 = \widehat{p}$ and we end up in the first case). By equation (4), $z_1 = x - \widehat{p} - (v - \widehat{p}) = x - v$, and by equation (2), $\widehat{p} = \frac{x - z_1}{2} = \frac{1}{2}v$. Also, by equation (4), $z_2 = x - z_1 - (u - z_1) = x - u$.

Case 2.2) Let the equilibrium price profile be (z_1, z_2). Then $z_1 = x - z_2 - (v - z_2) = x - v$, and $z_2 = x - z_1 - (u - z_1) = x - u$.

Step 2. Next we show that the following three statements are true.

(a) "If (a) is satisfied, case 2.0) is the unique equilibrium."

(Existence) First we show that case 2.0) is an equilibrium. This is justified by the fact that $z_1 - \widehat{p} = x - v - \frac{x}{3} = \frac{1}{3}(2x - 3v) \geq 0$ by (a). Also $z_2 > z_1 \geq \widehat{p}$.

(Uniqueness) Next we show that case 2.0) is the unique equilibrium. Assume by absurdity that case 2.1) occurs. By equation (3), $r_1 = \frac{1}{2}(x - \widehat{p}) = \frac{1}{4}(2x - v)$. Then $r_1 - z_1 = \frac{1}{4}(3v - 2x) \leq 0$ by (a). Hence pool 1 is not strictly binded by the competition margin ($r_1 > z_1$), contradicting our assumption. Assume by absurdity that case 2.2) occurs. By equation (3), $r_2 = \frac{1}{2}(x - z_1) = \frac{1}{2}v$. Then $r_2 - z_2 = \frac{1}{2}(2u + v - 2x) < \frac{1}{2}(3v - 2x) \leq 0$ by (a). Hence pool 2 is binded by the demand margin, contradicting our assumption.

(b) "If (b) is satisfied, case 2.1) is the unique equilibrium."

(Existence) By equation (3), $r_1 = \frac{1}{2}(x - \widehat{p}) = \frac{1}{4}(2x - v) > z_1 = x - v$, by $x < \frac{3}{2}v$. Also, $\widehat{p} = \frac{1}{2}v \leq z_2 = x - u$, by $x \geq u + \frac{1}{2}v$.

(Uniqueness) Assume by absurdity that case 2.0) occurs. Then $\widehat{p} - z_1 = \frac{x}{3} - (x - v) = \frac{1}{3}(3v - 2x) > 0$ by (b). Hence pool 1 is strictly binded by the competition margin, contradicting our assumption. Assume by absurdity that case 2.2) occurs. Then by equation (3), $r_2 = \frac{1}{2}(x - z_1) = \frac{1}{2}(x - (x - v)) = \frac{1}{2}v$. Then $r_2 - z_2 = \frac{1}{2}v - (x - u) = \frac{1}{2}(2u + v - 2x) \leq 0$ by (b). Hence pool 2 is binded by the demand margin, contradicting our assumption.

(c) "If (c) is satisfied, case 2.2) is the unique equilibrium."

(Existence) By equation (3), $r_2 = \frac{1}{2}(x - z_1) = \frac{1}{2}v > z_2 = x - u$, by (c). Also, $r_1 - z_1 = \frac{1}{2}(x - z_2) - z_1 = \frac{1}{2}(u - 2x + 2v) > \frac{1}{2}(2u - 2x + v) > 0$ by (c).

(Uniqueness) Assume by absurdity that case 2.0) occurs. Then $\widehat{p} - z_1 = \frac{1}{3}(3v - 2x) > \frac{1}{3}(2u + v - 2x) > 0$ by (c). Hence pool 1 is strictly binded by the competition margin, contradicting our assumption. Assume by absurdity that case 2.1) occurs. Then by equation (3), $r_2 = \frac{1}{2}(x - z_1) = \frac{1}{2}(x - (x - v)) = \frac{1}{2}v$. Then $r_2 - z_2 = \frac{1}{2}v - (x - u) = \frac{1}{2}(2u + v - 2x) > 0$ by (c). Hence pool 2 is strictly binded by the competition margin, contradicting our assumption.

Step 3. The rest of the calculation is summarized in the following table:

If	(a) $x \geq \frac{3}{2}v$	(b) $u + \frac{1}{2}v \leq x < \frac{3}{2}v$	(c) $x < u + \frac{1}{2}v$
$\boldsymbol{p}$	$(\widehat{p})_2 = \left(\frac{x}{3}\right)_2$	$(z, \widehat{p}) = \left(x - v, \frac{v}{2}\right)$	$(z_1, z_2) = (x - v, x - u)$
$\mathbf{P}$	$\frac{2}{3}x$	$\frac{2x - v}{2}$	$2x - v - u$
$D(\mathbf{P})$	$\frac{1}{3\Delta}x$	$\frac{v}{2\Delta}$	$\frac{v + u - x}{\Delta}$
π	$\left(\frac{x^2}{9\Delta}, \frac{x^2}{18\Delta}\right)$	$\left(\frac{v}{2\Delta}(x - v), \frac{v^2}{8\Delta}\right)$	$\frac{v + u - x}{\Delta}\left(x - v, \frac{x - u}{2}\right)$.

$\square$

Proof of Proposition 2: *Step 1.* First notice that there are four possible cases in terms of the number m' of the pools strictly binded by the competition margin. Furthermore, we show that case 3.1) does not exist under any circumstances.

Case 3.0) Let the equilibrium price profile be $(\widehat{p})_3$. By equation (2), $\widehat{p} = \frac{x}{4}$, and by equation (4), $z = V(3) - 2\widehat{p} - \max_{k=1,2} \{V(k) - k\widehat{p}\}$.

Case 3.1) Let the equilibrium price profile be $(z, \widehat{p}, \widehat{p})$ with $z < \widehat{p}$. Then $\widehat{p} = \frac{1}{3}(x - z)$, and $z = V(3) - 2\widehat{p} - \max_{k=1,2} \{V(k) - k\widehat{p}\}$. Assume by absurdity that this case exists. Then $z_2 = x - z - \widehat{p} - \max\{u - z, v - z - \widehat{p}\}$. Then $z - z_2 = -\widehat{p} + z - \max_{k=1,2} \{V(k) + \overline{\theta} - k\widehat{p}\} + \max\{u - z, v - z - \widehat{p}\} = 0$, whenever $u > v - \widehat{p}$ or $u \leq v - \widehat{p}$. This implies that $z \geq \widehat{p}$, contradicting our assumption.

Case 3.2) Without loss of generality, let the equilibrium price profile be $(z_1, z_2, \widehat{p})$ with $z_1 \leq z_2 < \widehat{p}$. By equation (2), $\widehat{p} = \frac{1}{2}(x - z_1 - z_2)$, $z_1 = x - z_2 - \widehat{p} - \max\{u - z_2, v - z_2 - \widehat{p}\}$, and $z_2 = x - z_1 - \widehat{p} - \max\{u - z_1, v - z_1 - \widehat{p}\}$. (Notice that $u - \widehat{p} < u - z_2 \leq u - z_1$.) Assume by absurdity that $u \leq v - \widehat{p}$, then $z_1 = z_2 = x - v$, and $z_3 = x - z_1 - z_2 - \max\{u - z_1, v - z_1 - z_2\} = x - z_1 - z_2 - (v - z_1 - z_2) = x - v$. $z_2 = z_3$ implies that $\widehat{p} \leq z_2$, contradicting our assumption. Hence $u > v - \widehat{p}$, and $z_1 = z_2 = x - u - \widehat{p}$.

If $x \leq v + \widehat{p}$, then $u \leq v - z_2$ and $z_3 = x - v$. From $x - v \leq \widehat{p}$, we have $\widehat{p} \geq z_3$, and hence $\widehat{p} = z_3 = x - v$, $z_1 = z_2 = v - u$. By equation (2), $\widehat{p} = \frac{1}{2}(x - 2v + 2u)$. Hence it must be that $x = 2u$. Also, from $z_2 < \widehat{p}$, we have $v - u < x - v$ (strict convexity). If $x > v + \widehat{p}$, then $u > v - z_2$ and $z_3 = x - u - z_2 = \widehat{p}$. By equation (2), $\widehat{p} = \frac{1}{2}(x - z_1 - z_2)$. Hence it must be that $x = 2u$. Also, from $x - v > \widehat{p}$ and $z_2 > v - u$, we have $x - v > v - u$ (strict convexity). Therefore, if case 3.2) exists, $V(\cdot)$ must be strictly convex, $x = 2u$, and the equilibrium price profile is $(z, z, \widehat{p})$ with $z < \widehat{p} = z_3$.

Case 3.3) Without loss of generality, let the equilibrium price profile be (z_1, z_2, z_3) with $z_1 \leq z_2 \leq z_3$. Then $z_1 = x - z_2 - z_3 - \max\{u - z_2, v - z_2 - z_3\}$, $z_2 = x - z_1 - z_3 - \max\{u - z_1, v - z_1 - z_3\}$, and $z_3 = x - z_1 - z_2 - \max\{u - z_1, v - z_1 - z_2\}$. Obviously, $z_1 = z_2$, and let the equilibrium price profile be (z, z, z_3) with $z \leq z_3$.

Step 2. Next we show that the following three statements are true.

(d+f) "If (d) and (f) are satisfied, case 3.0) is the unique equilibrium."

(Existence) To show $\widehat{p} \leq z$, it suffices to show that $\widehat{p} \leq x - 2\widehat{p} - \{u - \widehat{p}\}$ and $\widehat{p} \leq x - 2\widehat{p} - \{v - 2\widehat{p}\}$, which are guaranteed by (d) and (f) respectively.

(Uniqueness) It is obvious that case 3.2) does not occur, since $x \neq 2u$. Assume by absurdity that case 3.3) occurs. If $u > v - z_3$, $z = x - u - z_3$. Then by equation (3) $r_3 = \frac{1}{2}(x - 2z) = \frac{1}{2}(x - 2(x - u - z_3)) = \frac{1}{2}(2u - x + 2z_3)$. From $r_3 > z_3$, we have $x < 2u$, contradicting (d). If $u \leq v - z_3 (\leq v - z)$, $z = z_3 = x - v$. Then by equation (3) $r_3 = \frac{1}{2}(x - 2z) = \frac{1}{2}(x - 2(x - v)) = \frac{1}{2}(2v - x)$. From $r_3 > z_3$, we have (g) contradicting (f).

(concavity+g) "If (concavity) and (g) are satisfied, case 3.3) with $z = z_3 = x - v$ is the

unique equilibrium."

(Existence) By Proposition A2, if case 3.3) occurs, $z = z_3 = x - v$ because of the concavity of $V(\cdot)$. Also, by equation (3), $r = \frac{1}{2}(x - 2z) = \frac{1}{2}(2v - x) > z = x - v$ by (g).

(Uniqueness) It is obvious that case 3.2) does not occur because of the concavity of $V(\cdot)$. Assume by absurdity that case 3.0) occurs. Then $z = x - 2\widehat{p} - \max_{k=1,2}\left\{V(k) + \overline{\theta} - k\widehat{p}\right\} \leq x - v < \widehat{p} = \frac{x}{4}$ by (g), contradicting our assumption.

(e+convexity) "If (e) and (convexity) are satisfied, every equilibrium price profile is in the form of $(z, z, x - u - z)$ with $z \in \left[v - u, \frac{x-u}{2}\right]$."

(Existence) $z_3 = x - 2z - \max_{k=1,2}\left\{V(k) + \overline{\theta} - kz\right\}$, and by $z \geq v - u$, $z_3 = x - 2z - \{u - z\} = x - u - z$. Also, by $z \leq \frac{x-u}{2} \overset{\text{(convexity)}}{\leq} x - v$, $u - z \geq v - (x - u)$ and hence $z_1 = x - (x - u) - \max\{u - z, v - (x - u)\} = z$. By equation (3), $r_1 = \frac{1}{2}(x - (x - u)) = \frac{1}{2}u \overset{\text{(e)}}{\geq} \frac{1}{2}(x - u) \geq z$. Also, $r_3 = \frac{1}{2}(x - 2z)$, and by (e), $r_3 - (x - u - z) = u - \frac{1}{2}x \geq 0$. [Notice that if inequality (e) is strictly satisfied, it is case 3.3). If (e) is satisfied in equality, it is case 3.0) if $z = \frac{x-u}{2}$, and case 3.2) otherwise.]

(Any other price profile cannot be the equilibrium.) Assume by absurdity that the equilibrium price profile is (p, p, p_3) with $p < v - u$. (Notice that in all the possible cases, pools 1 and 2 charge the same price.) If case 3.0) occurs, we have $p = p_3 = \frac{x}{4} < v - u \overset{\text{(convexity)}}{\leq} \frac{x-u}{2}$, and hence $x > 2u$, contradicting (e). If case 3.2) or 3.3) occurs, by $p < v - u$, $p_3 = x - 2p - \max_{k=1,2}\left\{V(k) + \overline{\theta} - kp\right\} = x - v$. Then $p = x - p - p_3 - \max\{u - p, v - p - p_3\} = v - p - \max\{u - p, 2v - x - p\} \overset{\text{(convexity)}}{=} v - p - (u - p) = v - u$, contradicting our assumption of $p < v - u$.

Assume by absurdity that the equilibrium price profile is (p, p, p_3) with $p > \frac{x-u}{2}$. If case 3.0) occurs, we have $p = p_3 = \frac{x}{4} > \frac{x-u}{2} \overset{\text{(convexity)}}{\geq} v - u$ and hence $z = x - 2p - \max_{k=1,2}\left\{V(k) + \overline{\theta} - kp\right\} = x - u - p < x - u - \frac{x-u}{2} = \frac{x-u}{2} < p$, contradicting our assumption of $z \geq p$. If case 3.2) or 3.3) occurs, by $p > \frac{x-u}{2} \geq v - u$, $p_3 = x - 2p - \max_{k=1,2}\left\{V(k) + \overline{\theta} - kp\right\} = x - u - p < \frac{x-u}{2} < p$, contradicting our assumption of $p_3 \geq p$.

Step 3. The rest of the calculation is summarized in the following table:

If	(d) $x > 2u$, (f) $x \geq \frac{4}{3}v$	(concavity) $x < 2v - u$ (g) $x < \frac{4}{3}v$	(e) $x \leq 2u$, (convexity) $x \geq 2v - u$
$\boldsymbol{p}$	$(\widehat{p})_3 = \left(\frac{x}{4}\right)_3$	$(z)_3 = (x - v)_3$	$(z, z, x - u - z), z \in \left[v - u, \frac{x-u}{2}\right]$
$\mathbf{P}$	$\frac{3}{4}x$	$3(x - v)$	$x - u + z$
$D(\mathbf{P})$	$\frac{1}{4\Delta}x$	$\frac{3v-2x}{\Delta}$	$\frac{u-z}{\Delta}$
π	$\left(\frac{1}{16\Delta}x^2\right)_3$	$\left(\frac{3v-2x}{\Delta}(x - v)\right)_3$	$\frac{u-z}{\Delta}(z, z, x - u - z)$.

□

Proof of Proposition 3. It suffices to show the sign of $\frac{x^2}{12\Delta} - \frac{v}{2\Delta}(x - v)$. Let $\frac{x^2}{12\Delta} -$

$\frac{v}{2\Delta}(x-v) \equiv \frac{1}{12\Delta}h_{b1}(x)$, where

$$h_{b1}(x) = x^2 - 6vx + 6v^2.$$

$\frac{dh_{b1}}{dx} = 2(x-3v) < 0$ by $x < \frac{3}{2}v$. Hence $h_{b1}(x)$ is strictly decreasing in $\left[\frac{4}{3}v, \frac{3}{2}v\right]$. Therefore, together with $h_{b1}\left(\frac{4}{3}v\right) < 0$, $h_{b1}(x) < 0$. $\square$

Proof of Proposition 4: (b+concavity+g) Step I. Let $\frac{v^2}{8\Delta} - \frac{3v-2x}{\Delta}(x-v) \equiv \frac{1}{8\Delta}g_{bg}(x)$, where

$$g_{bg}(x) = v^2 + 8(2x-3v)(x-v).$$

$g_{bg}(x)$ obtains its minimum at $x = \frac{5}{4}v$, and $g_{bg}(\frac{5}{4}v) = 0$. Hence $g_{bg}(x) \geq 0$ and $\{1,2\}$ is an EPS.

Step III. $(x < x_2)$ When $x < x_2$, $\frac{x^2}{12\Delta} < \frac{v^2}{8\Delta}$ and $\{2\}$ has the incentive to defect.

When $x \geq x_2$, we investigate the part of $\{1\}$. The solution to $h_{b1}(x) = 0$ is $x = x_3$.

$(x_2 \leq x \leq x_3)$ When $x \leq x_3$, $h_{b1}(x) \geq 0$. $(x > x_3)$ When $x > x_3$, $h_{b1}(x) < 0$.

(c+concavity+g) Step I. We need to check the sign of $\frac{1}{2\Delta}(v+u-x)(x-u) - \frac{1}{\Delta}(3v-2x)(x-v)$, which equals to $\frac{1}{2\Delta}(x-2v+u)(3x-3v-u)$. By concavity, $x-2v+u<0$. Next we show that $x \leq \frac{1}{3}u+v$, and hence $\{1,2\}$ is an EPS. Assume by absurdity that $x > \frac{1}{3}u+v$, then $\frac{1}{3}u+v < u+\frac{1}{2}v$ by (c), and $\frac{1}{3}u+v < 2v-u$ by concavity. The former inequality implies that $v < \frac{4}{3}u$, and the latter $v > \frac{4}{3}u$.

Step III. First we show that $\{1\}$ has no incentive to break away from the complete pool. Let $\frac{x^2}{12\Delta} - \frac{v+u-x}{\Delta}(x-v) \equiv \frac{1}{12\Delta}h_{c1}(x)$, where

$$h_{c1}(x) = 13x^2 - (24v+12u)x + 12v^2 + 12uv.$$

The solution to $h_{c1}(x) = 0$, when existing[41], is

$$x = \frac{1}{13}\left(6u + 12v \pm 2\sqrt{3\gamma}\right), \text{ where } \gamma \equiv -v^2 - uv + 3u^2$$

if $\gamma \geq 0$, i.e., $v \leq \frac{\sqrt{13}-1}{2}u$. Then it suffices to show that $\frac{1}{13}\left(6u+12v-2\sqrt{3\gamma}\right)$ is larger than all the x's satisfying (c+concavity+g). This is justified by the fact that

$$\frac{1}{13}\left(6u+12v-2\sqrt{3\gamma}\right) \geq 2v-u \Leftrightarrow 13(4v-5u)^2 \geq 0.$$

Next we investigate the part of $\{2\}$. Let $\frac{x^2}{12\Delta} - \frac{1}{2\Delta}(v+u-x)(x-u) \equiv \frac{1}{12\Delta}h_{c2}(x)$, where

$$h_{c2}(x) = 7x^2 - (6v+12u)x + 6uv + 6u^2.$$

[41] When the real solution to $h_{c1}(x) = 0$ does not exist, i.e., $\gamma < 0$, then $h_{c1}(x) > 0$ trivially.

The solution to $h_{c2}(x) = 0$ is $x = x_4$ or x_5, if $\delta \geq 0$, i.e., $v \geq \frac{\sqrt{7}+1}{3}u$.

($x_4 < x < x_5$) When $x_4 < x < x_5$, $h_{c2}(x) < 0$. ($x \leq x_4$ or $x \geq x_5$) Otherwise, $h_{c2}(x) \geq 0$. $\square$

Proof of Proposition 5: (a+e+convexity) First notice that to make the set of x's satisfying (a+e+convexity) nonempty, we must have $\frac{3}{2}v \leq 2u$ and $2v - u \leq 2u$, implying that $v \leq \frac{4}{3}u$. This, in turn, implies that $\frac{3}{2}v > 2v - u$. So (a+e+convexity) is equivalent to $\frac{3}{2}v \leq x \leq 2u$.

Step I. Let $\frac{x^2}{18\Delta} - \frac{(3u-x)(x-u)}{4\Delta} \equiv \frac{1}{36\Delta} g_{ae}(x)$, where

$$g_{ae}(x) \equiv 11x^2 - 36ux + 27u^2.$$

The solution to $g_{ae}(x) = 0$ is $x = \frac{18 \pm 3\sqrt{3}}{11}u$, one smaller than $\frac{3}{2}v$ and one larger than $2u$. So $g_{ae}(x) < 0$ and $\{1, 2\}$ is not an EPS. We proceed to step II.

(c+e+convexity) First notice that to make the set of x's satisfying (c+e+convexity) nonempty, we must have $2v - u \leq 2u$ and $2v - u < u + \frac{1}{2}v$, implying that $v < \frac{4}{3}u$.

Step I. $\frac{(v+u-x)(x-u)}{2\Delta} - \frac{(3u-x)(x-u)}{4\Delta} = \frac{1}{4\Delta}(2v - u - x)(x - u) \leq 0$ by convexity, with equality when $x = 2v - u$. Notice that when $x = 2v - u$,

$$\frac{(v+u-x)(x-v)}{\Delta} = \frac{(v+u-x)(x-u)}{2\Delta} = \frac{(3u-x)(x-u)}{4\Delta},$$

so it is sufficient to consider the case with $\frac{(v+u-x)(x-u)}{2\Delta} < \frac{(3u-x)(x-u)}{4\Delta}$. Then $\{1, 2\}$ is not an EPS, and we proceed to step II.

(b+e+convexity) First notice that when $\frac{4}{3}u \leq v \leq \frac{1}{\sqrt{3}-1}u$, x_3 and x_6 satisfy (b+e+convexity). This is guaranteed by

$$\begin{aligned} 2v - u &\leq x_6 \Leftrightarrow (3v - 4u)^2 \geq 0, u + \frac{1}{2}v \leq x_6 \Leftrightarrow v \geq \frac{4}{3}u, x_6 < \frac{3}{2}v \Leftarrow v \geq \frac{4}{3}u; \text{ and} \\ 2v - u &\leq x_3 \Leftrightarrow v \leq \frac{1}{\sqrt{3}-1}u, x_3 \leq 2u \Leftrightarrow v \leq \frac{2}{3-\sqrt{3}}u, u + \frac{1}{2}v \leq x_3 \Leftrightarrow v \geq \frac{1}{5/2-\sqrt{3}}u. \end{aligned}$$

Step I. Let $\frac{v^2}{8\Delta} - \frac{(3u-x)(x-u)}{4\Delta} \equiv \frac{1}{8\Delta} g_{be}(x)$, where

$$g_{be}(x) \equiv 2x^2 - 8ux + v^2 + 6u^2.$$

The solution to $g_{be}(x) = 0$ is $x = x_6$ if $2u^2 - v^2 > 0$, i.e., $v < \sqrt{2}u$.

($x > x_6$) When $x > x_6$, $g_{be}(x) < 0$ and we proceed to step II. ($x \leq x_6$) Otherwise we proceed to step III.

Step III. First we show that $\{2\}$ has no incentive to break away from the complete pool.

By the following contradicting facts

$$u+\frac{1}{2}v\leq\sqrt{\frac{3}{2}}v\Leftrightarrow v\geq\frac{1}{\sqrt{3/2}-1/2}u,\text{ and }2v-u\leq\sqrt{\frac{3}{2}}v\Leftrightarrow v\leq\frac{1}{2-\sqrt{3/2}}u,$$

we have $\sqrt{\frac{3}{2}}v<\max\left\{u+\frac{1}{2}v,2v-u\right\}$ and hence $x>\sqrt{\frac{3}{2}}v$. Then

$$\frac{x^2}{12\Delta}-\frac{v^2}{8\Delta}=\frac{1}{12\Delta}\left[x+\sqrt{\frac{3}{2}}v\right]\left[x-\sqrt{\frac{3}{2}}v\right]>0.$$

Next we investigate the part of $\{1\}$. ($x\leq x_3$) As before, when $x\leq x_3$, $h_{b1}(x)\geq 0$. ($x>x_3$) When $x>x_3$, $h_{b1}(x)<0$. □

Proof of Proposition 6: (First part) If $x>x_1$, only the cases (a/b+d+f) and (a/b/c+e+convexity) can happen. Then use Propositions 3 and 5. If (a), only the cases (a+d+f) and (a+e+convexity) can happen.

(Second part) If $v<\frac{4}{3}u$, only the cases (a+d+f), (c+concavity+g) and (a/b/c+e+convexity) can happen. Use Propositions 3 and 5, and the fact that if $v<\frac{4}{3}u$, then $x_6<u+\frac{1}{2}v$ and $g_{be}(x)<0$. So $\{1,2\}$ is not an EPS and $\{3\}$ is the coarsest EPS. Furthermore, if $v<\frac{\sqrt{7}+1}{3}u$, by the proof of Proposition 4 about the case (c+concavity+g), $\delta<0$ and $h_{c2}(x)>0$. Therefore the coarsest EPS is $\{3\}$. □

Proof of Proposition 7: (a/b+d+f) Algebra tells that the total price is the highest in the fragmented structure among all the three pool structures. By Proposition 3, $\{1,1,1\}$ is never the coarsest EPS.

(a+e+convexity) Algebra tells that the total price is the lowest in the complete pool, which is always the coarsest EPS by Proposition 5.

(b+e+convexity) When $v>\frac{4}{3}u$, $3u-v<u+\frac{1}{2}v$ and hence $x>3u-v$. Then algebra tells that the total price is the highest in the fragmented structure, which is never the coarsest EPS by Proposition 5.

(b+concavity+g) When $v>\frac{10}{7}u$, the proof is summarized below. H, L, M denote the high, low, and medium total price. The asterisk means that the pool structure in question is the coarsest EPS (under some condition which follows). When $\left(\frac{4}{3}u<\right)v\leq\frac{10}{7}u$, a similar method applies and is omitted.

	$\{3\}$	$\{1,2\}$	$\{1,1,1\}$
$x<\frac{6}{5}v\,(\Rightarrow x<x_2)$	M	H (*)	L
$x\in\left(\frac{6}{5}v,\frac{5}{4}v\right)$	L (*$x\geq x_2$)	H (*$x<x_2$)	M
$x>\frac{5}{4}v\,(\Rightarrow x>x_2)$	L (*)	M (*)	H

(c+e+convexity) It can be shown that when $x > \frac{3}{2}u$, the complete pool charges the lowest total price. Otherwise, the fragmented structure charges the lowest. By Proposition 5, the complete pool is always the coarsest EPS.

(c+concavity+g) A similar method applies as in (b+concavity+g) and is omitted. □

Proof of Proposition 8: It suffices to show that $\{A, aa\}$ is always an EPS in all the cases with (e+convexity) ($\Rightarrow \{a, a, A\}$ is never the coarsest EPS), and then (under some circumstance) the complete pool is blocked by $\{A, aa\}$.

(a+e+convexity) Let $\frac{x^2}{18\Delta} - \frac{(2u-v)(v-u)}{\Delta} \equiv \frac{1}{18\Delta} g_1(x)$, where

$$g_1(x) \equiv x^2 + 18v^2 - 54uv + 36u^2.$$

The solution to $g_1(x) = 0$ is

$$x = 3\sqrt{2\zeta}, \text{ where } \zeta \equiv (2u - v)(v - u) > 0 \text{ by (e)}.$$

Since $3\sqrt{2\zeta} \leq \frac{3}{2}v$,[42] $x > 3\sqrt{2\zeta}$ by (a) and hence $g_1(x) \geq 0$. So $\{A, aa\}$ is an EPS.

Next we check whether $\{aaA\}$ is blocked by $\{A, aa\}$. Obviously, A will defect.

(b+e+convexity) By the fact that

$$\frac{v^2}{8\Delta} - \frac{(2u-v)(v-u)}{\Delta} = \frac{1}{8\Delta}(3v - 4u)^2 \geq 0,$$

$\{A, aa\}$ is an EPS. Then by Proposition 5, A will defect from the complete pool if $x_3 < x < x_6$.

(c+e+convexity) By convexity, $2v - u - x \leq 0$. By $v < \frac{4}{3}u$, $3u - v > u + \frac{1}{2}v$, and hence $x < 3u - v$ by (c). Then

$$\frac{(v+u-x)(x-u)}{2\Delta} - \frac{(2u-v)(v-u)}{\Delta} = \frac{1}{2\Delta}(2v - u - x)(x + v - 3u) \geq 0.$$

So $\{A, aa\}$ is an EPS.

Next, from the proof of Proposition 4, we know that $x_5 \leq 2v - u$. Then by convexity, $x \geq x_5$ and hence $h_{c2}(x) \geq 0$. So $\{aa\}$ will not defect.

We still need to check the possibility of A's defection. Notice that when $x > \frac{1}{13}\left(6u + 12v - 2\sqrt{3\gamma}\right)$, $h_{c1}(x) < 0$ and A will defect from the complete pool. Therefore it suffices to show that it is possible that $\frac{1}{13}\left(6u + 12v - 2\sqrt{3\gamma}\right)$ satisfies (c+e+convexity), and

$$\frac{1}{13}\left(6u + 12v + 2\sqrt{3\gamma}\right) > u + \frac{1}{2}v.$$

[42] This is guaranteed by the fact that $3\sqrt{2\zeta} \leq \frac{3}{2}v \Leftrightarrow (3v - 4u)^2 \geq 0$.

These are justified by

$$\begin{aligned}
\frac{1}{13}\left(6u+12v-2\sqrt{3\gamma}\right) &< u+\frac{1}{2}v \Leftrightarrow v<\frac{4\sqrt{3}+10}{13}u,\\
\frac{1}{13}\left(6u+12v-2\sqrt{3\gamma}\right) &\leq 2u \Leftrightarrow \sqrt{3\gamma}\geq 0>6v-10u,\\
\frac{1}{13}\left(6u+12v-2\sqrt{3\gamma}\right) &\geq 2v-u \Leftrightarrow 13\left(4v-5u\right)^2\geq 0,\\
\frac{1}{13}\left(6u+12v+2\sqrt{3\gamma}\right) &> u+\frac{1}{2}v \Leftrightarrow v<\frac{4\sqrt{3}+10}{13}u. \ \square
\end{aligned}$$

Proof of Proposition 9: We introduce some notations. Let

$$\begin{aligned}
\alpha &\equiv x^2-3ux+3u^2(>0),\\
\beta &\equiv 9u^2-2x^2(>0 \text{ by (e)}),\\
\kappa &\equiv 2v^2-7uv+6u^2(\geq 0 \text{ if } v\leq \frac{3}{2}u),\\
\underline{z} &\equiv \frac{u}{2}-\frac{\sqrt{\beta}}{6}, \text{ and } \overline{z}\equiv \frac{x}{2}-\frac{\sqrt{3\alpha}}{3}.
\end{aligned}$$

Step 1. When $z\in(-\infty,\overline{z})$, $\frac{x^2}{12\Delta}<\frac{(u-z)(x-u-z)}{\Delta}$, and (6) holds. To make $(-\infty,\overline{z})\cap\left[v-u,\frac{x-u}{2}\right]$ nonempty, we must have $\overline{z}>v-u$, implying that

$$x\in\left(-6v+12u-2\sqrt{6\kappa},-6v+12u+2\sqrt{6\kappa}\right). \tag{9}$$

This is guaranteed by the fact that $\left[\frac{3}{2}v,2u\right]\subsetneq$(9), which is in turn guaranteed by $v<\frac{4}{3}u$ (c.f., Proof of Proposition 5). Also, we can show that $\overline{z}\leq\frac{x-u}{2}$.[43] Thus, $\{z|(6) \text{ holds}\}=[v-u,\overline{z})$ is nonempty.

Step 2. When $z\in\left(\underline{z},\frac{u}{2}+\frac{\sqrt{\beta}}{6}\right)$, $\frac{x^2}{18\Delta}<\frac{(u-z)z}{\Delta}$, and (5) holds. We can show that $v-u\leq\underline{z}<\overline{z}<\frac{u}{2}+\frac{\sqrt{\beta}}{6}$.[44] Hence

$$[v-u,\overline{z})\cap\left(\underline{z},\frac{u}{2}+\frac{\sqrt{\beta}}{6}\right)=(\underline{z},\overline{z}).$$

Step 3. Notice that $\{a,aA\}$ is not an EPS no matter whether (5) holds or not. When $z\in[v-u,\underline{z}]$, (5) does not hold and hence $\{A,aa\}$ is an EPS. Obviously, it is the coarsest one. When $z\in\left[\overline{z},\frac{x-u}{2}\right]\subset\left(\underline{z},\frac{u}{2}+\frac{\sqrt{\beta}}{6}\right)$, (5) holds and hence $\{A,aa\}$ is not an EPS. Thus, $\{aaA\}$ is the coarsest EPS since (6) does not hold. $\square$

[43] This inequality is equivalent to $(2x-3u)^2\geq 0$.

[44] The first inequality is guaranteed by (a) and the fact that $\frac{3}{2}v\geq 3\sqrt{2}\sqrt{-v^2+3uv-2u^2}$. The second is guaranteed by (a) and (e). And the last is guaranteed by (e).

Proof of Proposition 10: To make $x - u + z < \frac{x}{2}$, we need $z < \frac{2u-x}{2} \equiv z^*$. When $x < \frac{18}{11}u$, $z^* > \underline{z}$ and $(\underline{z}, z^*) \neq \varnothing$; otherwise, $z^* \leq \underline{z}$ and any $z \in (\underline{z}, \overline{z})$ is larger than z^*. (By Proof of Proposition 5, (a+e+convexity) is equivalent to $x \in \left[\frac{3}{2}v, 2u\right]$. Note that when $v < \frac{12}{11}u$, $\frac{18}{11}u \in \left[\frac{3}{2}v, 2u\right]$.) $\square$

References

Aoki, R., & Nagaoka, S. (2006). Formation of a Pool with Essential Patents. Mimeo, Hitotsubashi University.

Bloch, F. (1995). Endogenous structures of association in oligopolies. *RAND Journal of Economics*, 537-556.

Bloch, F. (1996). Sequential formation of coalitions in games with externalities and fixed payoff division. *Games and Economic Behavior*, 14(1), 90-123.

Bloch, F. (2002). Coalitions and networks in industrial organization. *The Manchester School*, 70(1), 36-55.

Bloch, F. (2003). Non-Cooperative Models of Coalition Formation in Games with Spillovers. In *The Endogenous Formation of Economic Coalitions* (pp. 35-79). Edward Elgar.

Brenner, S. (2009). Optimal formation rules for patent pools. *Economic Theory*, 40(3), 373-388.

Choi, J. P. (2010). Patent pools and cross-licensing in the shadow of patent litigation. *International Economic Review*, 51(2), 441-460.

Clarkson, G. (2003). Sharper Machetes for the Patent Thicket: Objective Criteria for Evaluating Patent Pools in the Shadow of Antitrust Enforcement. HBS Working Paper.

Diamantoudi, E., & Xue, L. (2007). Coalitions, agreements and efficiency. *Journal of Economic Theory*, 136(1), 105-125.

Greenlee, P. (2005). Endogenous formation of competitive research sharing joint ventures. *Journal of Industrial Economics*, 53(3), 355-391.

Guellec, D., & De La Potterie, B. V. P. (2007). *The economics of the European patent system*. Oxford University Press.

Kamien, M. I. (1992). Patent licensing. *Handbook of game theory with economic applications*, 1, 331-354.

Kim, S. H. (2004). Vertical structure and patent pools. *Review of Industrial Organization*, 25(3), 231-250.

Lerner, J., & Tirole, J. (2004). Efficient Patent Pools. *American Economic Review*, 94(3), 691-711.

Levy, G. (2004). A model of political parties. *Journal of Economic Theory*, 115(2), 250-277.

Maskin, E. (2003). Bargaining, Coalition and Externalities. Presidential address of the Econometric Society.

Morasch, K. (2000). Strategic alliances as Stackelberg cartels–concept and equilibrium alliance structure. *International Journal of Industrial Organization*, 18(2), 257-282.

Quint, D. (2014). Pooling with essential and nonessential patents. *American Economic Journal: Microeconomics*, 6(1), 23-57.

Ray, D., & Vohra, R. (1997). Equilibrium binding agreements. *Journal of Economic Theory*, 73(1), 30-78.

Ray, D., & Vohra, R. (1999). A theory of endogenous coalition structures. *Games and Economic Behavior*, 26(2), 286-336.

Ray, D., & Vohra, R. (2001). Coalitional power and public goods. *Journal of Political Economy*, 109(6), 1355-1385.

Ray, D. (2007). *A game-theoretic perspective on coalition formation*. Oxford University Press.

Salant, S. W., Switzer, S., & Reynolds, R. J. (1983). Losses from horizontal merger: the effects of an exogenous change in industry structure on Cournot-Nash equilibrium. *Quarterly Journal of Economics*, 98(2), 185-199.

Scotchmer, S. (1991). Standing on the shoulders of giants: cumulative research and the patent law. *Journal of Economic Perspectives*, 5(1), 29-41.

Sen, D., & Tauman, Y. (2007). General licensing schemes for a cost-reducing innovation. *Games and Economic Behavior*, 59(1), 163-186.

Shapiro, C. (2001). Navigating the patent thicket: Cross licenses, patent pools, and standard setting. In *Innovation Policy and the Economy, Volume 1* (pp. 119-150). MIT press.

Taylor, C. T., Silberston, A., & Silberston, Z. A. (1973). *The economic impact of the patent system: a study of the British experience (Vol. 23)*. CUP Archive.

Von Neumann, J., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

Yi, S. S. (1998). Endogenous formation of joint ventures with efficiency gains. *RAND Journal of Economics*, 610-631.

Yi, S. S. (2003). Endogenous formation of economic coalitions: a survey of the partition function approach. In *Endogenous Formation of Economic Coalitions* (pp. 80-127). Edward Elgar.

Yi, S. S., & Shin, H. (2000). Endogenous formation of research coalitions with spillovers. *International Journal of Industrial Organization*, 18(2), 229-256.